The Testable Implications of Zero-sum Games

SangMok Lee*

November 14, 2011

Abstract

We study collective choices from the revealed preference theory viewpoint. For every product set of individual actions, joint choices are called Nash-rationalizable if there exists a preference relation for each player such that the selected joint actions are Nash equilibria of the corresponding game. We characterize Nash-rationalizable joint choice behavior by zero-sum games, or games of conflicting interests. If the joint choice behavior forms a product subset, the behavior is called interchangeable. We prove that interchangeability is the only additional empirical condition which distinguishes zero-sum games from general non-cooperative games.

JEL Classification Numbers: C72, D70

Keywords: Interchangeability, Nash-rationalizability, Revealed preference, Zero-sum game

1 Introduction

Suppose two players choose joint actions from a finite set of alternatives. As outside observers, we witness the joint choice behavior, but we may not know the exact payoffs leading the players to such group choices. By only observing joint choice behavior, we may ask whether people play Nash equilibrium, and, if they do, what type of games they play.

This paper derives falsifiable conditions of joint choice behavior from equilibrium play of a zero-sum game, or a game of conflicting interests. That is, we study additional behavioral implications of a game being zero-sum, in addition to the hypothesis of Nash equilibrium play. Instead of assuming a specific pattern of joint behavior, this study requires only weak

*Division of the Humanities and Social Sciences, 228-77, California Institute of Technology, Pasadena, CA, 91125. Email: `sangmok-at-hss.caltech.edu`. I am grateful to Christopher Chambers and Federico Echenique for encouragement and guidance. I also thank Luke Boosey, Guilherme Freitas, Ruth Mendel, Burkhard C. Schipper, Colin Stewart, Leeat Yariv, participants at the 20th International Conference on Game Theory, and two anonymous referees for their valuable comments.

rationality axioms: complete and transitive preferences at the individual level, and Nash equilibrium play at the collective level.

The main motivation of this exercise is that we want to be able to refute the notion that two agents are in "direct competition," and detect whether or not there could be "gains from cooperation," without knowing the exact payoffs. However, its applications are not limited to cases where we only observe joint choice behavior. Even when we observe the exact monetary returns (e.g., a laboratory experiment), the observed monetary returns may differ from utility payoffs which players perceive. For example, if each subject cares about her monetary return relative to her opponent's return, the joint behavior may follow Nash equilibrium behavior of a zero-sum game rather than the original game. This is because a two-person game with symmetric monetary returns becomes a symmetric zero-sum game with respect to relative monetary returns.[1] Based on the observed joint choice behavior, we can test whether subjects play the original game or the zero-sum game induced by their relative monetary returns.

Sprumont (2000) assumes that econometricians are given a choice correspondence defined on product sets of individual actions. The question is when the observed joint behavior is consistent with Nash equilibrium play, assuming players are rational and they play games simultaneously. Sprumont proves that the observed joint behavior is Nash rationalizable if and only if it satisfies *Persistent under Restriction* and *Persistent under Expansion* axioms, which are similar to classical axioms of choice theory (see, e.g., Moulin (1985)). We retain Sprumont's basic abstract setup and ask, "Is the choice correspondence Nash-rationalizable with a certain game category, specifically, zero-sum games?"

As an introductory example, Figure 1 shows how Nash-rationalizable choice behavior may not be rationalizable by a zero-sum game. In this example, player 1 conceivably choose either U or D and player 2 may choose L or R. Following classical choice theory, we may observe how players choose when choice sets are restricted. Figure 1 shows all the possible product subsets of $\{U, D\} \times \{L, R\}$ from which two players choose their joint actions. For each product subset, $(*)$ is the action profile chosen by the players. We can verify that the joint choice behavior exhibited in Figure 1 is consistent with Nash equilibrium behavior of a coordination game in which coordinating to (U, L) or (D, R) gives a higher payoff to both players.

This choice correspondence, however, does not follow Nash equilibrium behavior for any zero-sum game. We observe that (U, L) is chosen from $\{(U, L), (D, L)\}$ and (D, R) is chosen from $\{(D, L), (D, R)\}$. Assuming that the choices are Nash equilibria of a zero-sum game, these choices imply that for player 1, (U, L) is preferred to (D, L); for player 2, (D, R) is

[1] See, e.g., Duersch, Oechssler, and Schipper (2011).

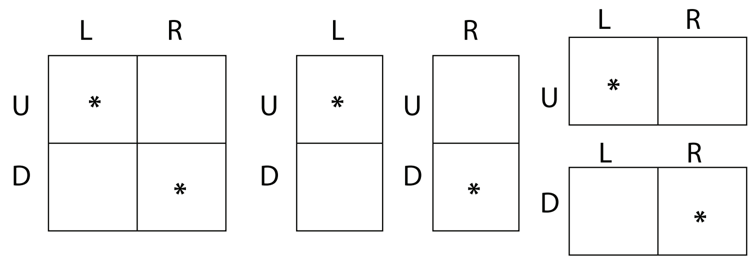

Figure 1: Nash-rationalizable but *not* by zero-sum games

preferred to (D, L), which indicates player 1 prefers (D, L) to (D, R). On the other hand, (D, R) is chosen from $\{(D, R), (U, R)\}$ and (U, L) is chosen from $\{(U, L), (U, R)\}$. For player 1, (D, R) is preferred to (U, R); for player 2, (U, L) is preferred to (U, R), which implies player 1 prefers (U, R) to (U, L). As a result, the preference of player 1 forms a cycle, which implies that all possible joint actions are indeed indifferent for player 1 (and thus for player 2 by the fact that the game is zero-sum). Therefore, we would expect to see all strategy profiles chosen.

This example shows that once we have two choices on the diagonal in a table of joint actions, the other two pairs of actions must also be chosen in order for the joint choices to follow Nash equilibrium behavior for a zero-sum game. When choice behavior forms a product subset for each game table, we say that the choice behavior is *interchangeable*. Although it is easy to identify that interchangeability is necessary, whether the condition is sufficient is not as straightforward.

Our main theorem shows that this interchangeability of joint choice behavior is indeed the only additional condition that distinguishes the testable implications of zero-sum games from those of general non-cooperative games. It is worth pointing out two assumptions behind the theorem. First, we restrict Nash rationalizability to pure strategy Nash equilibria. Second, we assume complete observations, where choices are observed from all product sets of individual actions.

This paper follows a broad range of revealed preference theory. Since Samuelson (1938), there have been numerous papers on revealed preference theory in various settings. In the context of collective choice, Wilson (1970) and Plott (1974) study cooperative games and find that the Weak Axiom implies the solution concept proposed by von Neumann and Morgenstern. More recently, Echenique and Ivanov (2011) and Chambers and Echenique (2011) study the testable implications of collective decision making such as household behavior and bargaining over money.

The testable implications of game theoretic models have grown only recently relative to the history and popularity of game theory. Peleg and Tijs (1996) and Sprumont (2000) find conditions of joint choice behaviors being consistent with Nash equilibria, as the corresponding games are reduced or expanded. Galambos (2009) weakens the complete observation assumption, and Demuynck and Lauwers (2009) study joint choices over lotteries. The two approaches adopt Richter (1971)'s congruence axiom, and find that the modified versions of the congruence axiom are necessary and sufficient conditions for Nash rationalizability. Ray and Zhou (2001), and Ray and Snyder (2003) consider extensive form games, and find conditions such that sequential choices are rationalizable by a subgame perfect Nash equilibrium. Xu and Zhou (2007) characterize conditions under which choices are rationalizable by game trees when the choice process is not observable.

In the context of more concrete games, Forges and Minelli (2009) apply their main result to market games, in which each player's budget constraint depends on other players' actions. For the model of Cournot competition, Carvajal, Deb, Fenske, and Quah (2010) consider the case of observing a finite set of prices and quantities, and Cherchye, Demuynck, and De Rock (2011) consider the case of observing price and quantity functions defined over exogenous variables. Both studies characterize conditions under which their observed data are consistent with the model of Cournot competition.

2 Model and Main Theorem

There are two players, 1 and 2. Let A_1 and A_2 be finite sets of actions that players 1 and 2 may conceivably choose. $A := A_1 \times A_2$ is the set of all possible joint actions. Following the classical revealed preference approach, suppose we observe choices from $B := B_1 \times B_2$ in which $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$ are the sets of available actions for player 1 and 2. In this model, all choices from each $B \subseteq A$ can be summarized as a choice correspondence.

Definition 1 *Let $\mathcal{A} := \{B = B_1 \times B_2 | \emptyset \neq B \subseteq A\}$ be the set of all nonempty product sets included in A. A **joint choice correspondence** f assigns to each $B \in \mathcal{A}$ a nonempty set $f(B) \subseteq B$.*

In the case where at most one player has more than one available action in B, we say that B is a *line*. Depending on the player, the line is either in a *column* or a *row* - the former when player 1 has choices, the latter when player 2 has choices. In addition, we call a product subset $B \in \mathcal{A}$ a *feasible set*. For any $B'' \subseteq B$ and $B'' \in \mathcal{A}$, we call B'' a *feasible subset* of B. For any $B, B' \in \mathcal{A}$, define $B \vee B'$ as the set of all possible pairs of actions from B_i and B_i' $(i = 1, 2)$. That is,

$$B \vee B^{'} := \prod_{i=1,2} (B_i \cup B_i^{'})$$

Suppose we wish to test whether a choice correspondence is rational or not. First, we shall assume that each player is individually rational. That is, each player has a preference relation over joint actions, and these relations are complete and transitive.[2] We call such relations **weak orders**. In addition, we wish to test if the players are collectively rational. In terms of collectively rationality, we assume that players play a Nash equilibrium. The following definition is our notion of rationalizability of collective choice behavior.

Definition 2 *A joint choice correspondence f is* ***Nash-rationalizable*** *if there are two weak orders $\succeq_1, \succeq_2$ on A such that for each $B \in \mathcal{A}$, $f(B)$ coincides with the set of all Nash equilibria of the game $(B, \succeq_1, \succeq_2)$.*[3]

Sprumont (2000) introduces the following conditions for Nash-rationalizability. These conditions are extended versions of *Sen's* α, β, and γ in individual choice theory (see, e.g., Moulin (1985)).[4] When a feasible set is restricted to a line, the first condition coincides with *Sen's* α and β, and the second condition coincides with *Sen's* γ.

Definition 3 *A joint choice correspondence over $\mathcal{A}$ is:*

- ***Persistent under Contraction*** *(PC):*

  (PC1) : For all $B, B^{'} \in \mathcal{A}$ with $B^{'} \subseteq B$, $f(B) \cap B^{'} \subseteq f(B^{'})$.

  (PC2) : Moreover, if B is a line, $B' \subseteq B$ and $f(B) \cap B^{'} \neq \emptyset$ implies $f(B^{'}) \subseteq f(B)$.

- ***Persistent under Expansion*** *(PE): For all $B, B^{'} \in \mathcal{A}$, $f(B) \cap f(B^{'}) \subseteq f(B \vee B^{'})$.*

With these two conditions, Sprumont (2000) establishes the following theorem.

Theorem 4 *A joint choice correspondence f is Nash-rationalizable if and only if it satisfies (PC) and (PE).*

Using this model of Nash-rationalizability, we restrict the set of available rationalizing games from the set of all non-cooperative games to include only zero-sum games, or games of conflicting interests. Under the conditions of zero-sum games, the preferences of two players

[2] A relation $\succeq$ is called *complete* if for all joint choices $a, b \in A$, it follows that $a \succeq b$ or $b \succeq a$, and is called *transitive* if for all $a, b, c \in A$ for which $a \succeq b$ and $b \succeq c$, it follows that $a \succeq c$.

[3] In other words, if $(b_1, b_2) \in f(B)$, then $(b_1, b_2) \succeq_1 (b_1', b_2)$ and $(b_1, b_2) \succeq_2 (b_1, b_2')$ for every $(b_1', b_2) \in B$ and $(b_1, b_2') \in B$.

[4] Although Moulin (1985) calls these conditions *Chernoff* and *Expansion*, *Sen's* α, β, and γ are more conventional terminologies in individual choice theory. See, for example, Austen-Smith and Banks (1994).

are opposed. Therefore, while a general non-cooperative game consists of two weak orders, zero-sum games require only a single weak order.

Definition 5 *Let $\succeq$ be a weak order over A, and $\preceq$ is the inverse relation of $\succeq$.*[5] *The game defined by $(A, \succeq, \preceq)$ is called* ***a two-person zero-sum game****. We say that a joint choice correspondence f is* ***Nash-rationalizable by a zero-sum game*** *if there is a weak order $\succeq$ on A such that for each $B \in \mathcal{A}$, $f(B)$ coincides with the set of all Nash equilibria of the game $(B, \succeq, \preceq)$.*

As demonstrated in Example 1, not all Nash-rationalizable joint choice correspondences are Nash-rationalizable by a zero-sum game. In the example, we needed one additional condition to fill the gap in the product space of the two distinct choices. We formally state this condition in the following definition.

Definition 6 (Interchangeability (INT)) *A joint choice correspondence f over $\mathcal{A}$ is* ***interchangeable*** *if for all $B \in \mathcal{A}$ and all $b, b' \in f(B)$, $\{b\} \vee \{b'\} \subseteq f(B)$.*

It is well-known that any pair of equilibrium strategies of a zero-sum game, one for each player, is an equilibrium strategy profile (see, e.g., Luce and Raiffa (1989)). Provided that players face a zero-sum game, and observed joint actions follow the Nash-equilibria of the corresponding games, the choice correspondence must be interchangeable. Our contribution is showing that interchangeability is indeed the only additional behavioral implication which distinguishes zero-sum games from general non-cooperative games. We summarize this result as the following main theorem.

Theorem 7 *A joint choice correspondence is Nash-rationalizable by a zero-sum game if and only if it satisfies (PC), (PE), and (INT).*

3 Discussion

Our model assumes the existence of a joint choice for all $B \in \mathcal{A}$. Accordingly, verifying whether a joint choice correspondence is Nash-rationalizable means assuming that all feasible sets have a pure strategy Nash equilibrium. Although a small literature provides conditions of zero-sum games having pure strategy Nash equilibria (Shapley, 1964; Radzik, 1991;

[5]Let $\succeq$ be a binary relation over A. We define the inverse relation $\preceq$ as

$$\text{for all } a, b \in A \text{ for which } a \succeq b,\ b \preceq a.$$

The inverse relation of a weak order is also a weak order. The proof is immediate by definition.

Duersch, Oechssler, and Schipper, forthcoming), the characterization of conditions that are both necessary and sufficient remains an open question. We may avoid this existence issue by investigating either mixed strategies or correlated strategies. However, these strategies introduce other difficulties since observed joint choices do not directly represent underlying preferences.

Our model also requires observed choices from all feasible sets. We may weaken this requirement by assuming *incomplete observations*, where a choice correspondence is defined on $\mathcal{A}^{'} \subseteq \mathcal{A}$. In classical choice theory, Richter (1971) shows that a choice correspondence with incomplete observations is rationalizable by a weak order if and only if it is congruent. Galambos (2009) generalizes Richter's congruence condition, and shows that the generalized congruence condition is necessary and sufficient for Nash-rationalizability with incomplete observations.

Unfortunately, interchangeability together with individual-level congruent choices is not sufficient for Nash-rationalizability by a zero-sum game. For example, suppose $\mathcal{A}^{'}$ is the set of $B := \{U, M\} \times \{L, C, R\}$, $B' := \{U, M, D\} \times \{C, R\}$, and all lines in B and B'. Suppose $f(B) = \{(M, R)\}$ and $f(B') = \{(U, C)\}$, and assume that choices in each line satisfy (PE) and (PC). The choices are congruent in terms of Galambos (2009), and therefore, Nash-rationalizable. However, the choices are not Nash-rationalizable by a zero-sum game. From $\{U, M\} \times \{C, R\}$, (U, C) and (M, R) are the only choices consistent with the choices in each line, but this observation violates interchangeability.[6]

A Proof of the Main Theorem

The necessity of (PC), (PE), and (INT) Suppose a joint choice correspondence f is Nash-rationalizable by a zero-sum game $(A, \succeq, \preceq)$. The necessity of (PC) and (PE) is obvious from the definition of Nash equilibrium. To show the necessity of (INT), let $B = B_1 \times B_2 \in \mathcal{A}$

[6] Alternatively, we may consider a *congruent* joint choice correspondence $f : \mathcal{A}' \rightrightarrows A$, where $\mathcal{A}' \subseteq \mathcal{A}$ is a set of observed games. We define a binary relation $\succeq$ on A by: for $a = (a_1, a_2), b = (b_1, b_2) \in A$,

$$\begin{aligned} a \succeq b \text{ if and only if there exists } B \in \mathcal{A}' \text{ such that } a, b \in B, \text{ and} \\ \text{either } a_2 = b_2 \text{ and } a \in f(B), \text{ or } a_1 = b_1 \text{ and } b \in f(B). \end{aligned}$$

If there is a finite sequence $c, d, \ldots, e$ such that $a \succeq c \succeq d \cdots \succeq e \succeq b$, then we write $a \, T_{\succeq} \, b$. We say that a joint choice correspondence f is *congruent*, if for all $a, b \in A$ and all $B \in \mathcal{A}'$,

$$a \, T_{\succeq} \, b, \; a \in B, \text{ and } b \in f(B) \implies a \in f(B).$$

Assuming that a joint choice correspondence is congruent is, however, almost the same as assuming its Nash-rationalizability by a zero-sum game. In particular, when $\mathcal{A}' = \mathcal{A}$, the assumption implies that the relation $\succeq$ is *consistent* (see Definition 8). Most of the proof in this paper is devoted to showing that $\succeq$ is consistent (see Section A.1).

and $b = (b_1, b_2), b' = (b'_1, b'_2) \in f(B)$. Note that $b_1, b'_1 \in B_1$ and $b_2, b'_2 \in B_2$, which implies that (b_1, b'_2) and (b'_1, b_2) are also in B.

Since (b_1, b_2) is a Nash equilibrium of the game $(B, \succeq, \preceq)$,

i) player 1 prefers (b_1, b_2) to (b'_1, b_2): i.e. $(b_1, b_2) \succeq (b'_1, b_2)$, and

ii) player 2 prefers (b_1, b_2) to (b_1, b'_2): i.e. $(b_1, b_2) \preceq (b_1, b'_2)$, or equivalently $(b_1, b'_2) \succeq (b_1, b_2)$.

In addition, since (b'_1, b'_2) is a Nash equilibrium of the game $(B, \succeq, \preceq)$,

iii) player 1 prefers (b'_1, b'_2) to (b_1, b'_2): i.e. $(b'_1, b'_2) \succeq (b_1, b'_2)$, and

iv) player 2 prefers (b'_1, b'_2) to (b'_1, b_2): i.e. $(b'_1, b'_2) \preceq (b'_1, b_2)$, or equivalently $(b'_1, b_2) \succeq (b'_1, b'_2)$.

By transitivity of $\succeq$, from (i) and (iv) we obtain $(b_1, b_2) \succeq (b'_1, b_2) \succeq (b'_1, b'_2)$, and from (ii) and (iii) we obtain $(b'_1, b'_2) \succeq (b_1, b'_2) \succeq (b_1, b_2)$. Therefore, (b_1, b_2), (b'_1, b_2), (b_1, b'_2), and (b'_1, b'_2) are all indifferent for player 1 and player 2.

In this situation, (b'_1, b_2) is a Nash equilibrium of the game $(B, \succeq, \preceq)$: for any $b''_1 \in B_1$, since (b_1, b_2) is a Nash equilibrium, we have $(b_1, b_2) \succeq (b''_1, b_2)$, and thus $(b'_1, b_2) \succeq (b''_1, b_2)$; from player 2's viewpoint, for any $b''_2 \in B_2$, since (b'_1, b'_2) is a Nash equilibrium, we have $(b'_1, b'_2) \preceq (b'_1, b''_2)$, and thus $(b'_1, b_2) \preceq (b'_1, b''_2)$. Similarly, (b_1, b'_2) is also a Nash equilibrium of the game $(B, \succeq, \preceq)$. In all, $\{b\} \vee \{b'\}$ is a subset of the set of Nash equilibria of the game $(B, \succeq, \preceq)$, and therefore a subset of $f(B)$.

The sufficiency of (PC), (PE), and (INT) To prove sufficiency, we construct a preference $\succeq$ over A, with which for all $B \in \mathcal{A}$, $f(B)$ coincides with the set of all Nash equilibria of $(B, \succeq, \preceq)$.

In individual choice theory, given a finite alternative set X and a choice correspondence g, Sen (1971) defines *base relation* R^* as

$$xR^*y \text{ if and only if } x \in g(\{x, y\}).$$

Similarly, we define two relations $\succeq^*$ and $\succeq^{**}$ as follows: for any $a = (a_1, a_2), b = (b_1, b_2) \in A$,

$$\begin{aligned} a \succeq^* b \quad &\text{if and only if} \quad a_2 = b_2 \text{ and } a \in f(\{a_1, b_1\} \times \{a_2\}), \\ a \succeq^{**} b \quad &\text{if and only if} \quad a_1 = b_1 \text{ and } b \in f(\{a_1\} \times \{a_2, b_2\}) \end{aligned}$$

Note that $\succeq^*$ and $\succeq^{**}$ are disjoint, and $\succeq^{**}$ is defined "inversely" from the convention of individual choice theory. Finally, let $\succeq$ be the union of $\succeq^*$ and $\succeq^{**}$. We arrange player 1's conceivable actions in a column and player 2's actions in a row, thereby constructing a table of joint actions. Then, in each line (PC) is equivalent to Sen's α and β, and $\succeq^*$ and $\succeq^{**}$ are defined as analogous with the base relation. $\succeq^*$ represents the base relation in each column, and $\succeq^{**}$ represents the base relation in each row, except $\succeq^{**}$ is defined inversely. In such case, Sen (1971) shows that $\succeq^*$ is a weak order in each column, and $\succeq^{**}$ is an inverse relation of a weak order in each row; therefore, the union $\succeq$ is a weak order in both columns and rows. Note that $\succeq$ is not yet defined on any pair of joint actions across the lines. In order to construct a complete relation over A, we need some preliminary definitions.

Definition 8 (Consistency) *Let R be a relation over $X = \{x^1, x^2, \ldots, x^l, \ldots\}$ and P be the strict counterpart of R. A sequence $x^1 R x^2 R \cdots R x^l P x^1$ is called a PR-cycle (or a cycle). If a relation does not have any cycle, we say that it is **consistent**.*

Definition 9 (Extension) *Given any arbitrary binary relation R over X, if a binary relation R' over X is such that*

$$xRy \text{ implies } xR'y$$
$$xPy \text{ implies } xP'y$$

*then R' is called an **extension** of R.*

In the following proof, we show using interchangeability that $\succeq$ is consistent (Section A.1). Then, we show using (PE) and (PC) that any weak order extension of $\succeq$ Nash-rationalizes the joint choice correspondence by a zero-sum game (Section A.2).

A.1 $\succeq$ is consistent.

By means of contradiction, suppose that there exists $\{a^1, \cdots, a^N\} \subseteq A$ such that $a^1 \succeq a^2 \succeq \cdots \succeq a^N \succ a^1$. Since $\succeq$ is the union of two disjoint sets, $\succeq^*$ and $\succeq^{**}$, $\succeq$ is either $\succeq^*$ or $\succeq^{**}$ depending on whether $\{a^i, a^j\}$ is in a column or a row.

Hereafter, we restrict our attention to cycles of an even length of at least 4 where the links in the cycle alternate between $\succeq^{**}$ and $\succeq^*$. This restriction does not lead to a loss of generality. First, we only need to consider cycles that alternate because any cycle containing consecutive $\succeq^*$ or $\succeq^{**}$ can be reduced by means of transitivity to a shorter cycle without consecutive $\succeq^*$ or $\succeq^{**}$. In addition, there is no cycle with a length of 2 such as $a^1 \succeq^* a^2 \succ^{**} a^1$. By definition of $\succeq^*$, $a^1_2 = a^2_2$, and by definition of $\succeq^{**}$, $a^1_1 = a^2_1$, which together imply that $a^1 = a^2$. Then, we have $a^1 \succ^{**} a^1$. We can also rule out cycles of odd lengths, since we can shorten any cycle of a odd length by transitivity to a cycle of an even length. For instance, the cycle $a \succeq^{**} b \succeq^{*} c \succeq^{**} d \succeq^{*} e \succ^{**} a$ of length 5 can be reduced to the cycle $b \succeq^{*} c \succeq^{**} d \succeq^{*} e \succ^{**} b$ of length 4. We also restrict attention to the case where the cycle begins with $\succeq^{**}$. The case where the cycle begins with $\succeq^{*}$ is omitted, but can be proved in a similar way.

First, we prove that there is no cycle of length 4. Suppose $a \succeq^{**} b \succeq^{*} c \succeq^{**} d \succ^{*} a$. By definition, we have $a_1 = b_1$, $b_2 = c_2$, $c_1 = d_1$ and $d_2 = a_2$. Then $\{a, b, c, d\}$ makes feasible sets as depicted in Figure 2. In part (i) of the figure, each dashed arrow corresponds to either $\succeq^{*}$ or $\succeq^{**}$ and the solid arrow corresponds to $d \succ^{*} a$. The tail of each arrow is the element from the left hand side of the preference relation.

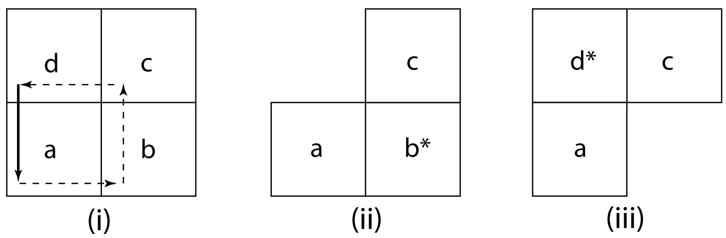

Figure 2: A cycle of length 4

Parts (ii) and (iii) of Figure 2 illustrate the choice correspondence generating $\succeq^{*}$ and $\succeq^{**}$ for each feasible set. Note that $b \in f(\{a, b\}) \cap f(\{b, c\})$, and $d \in f(\{a, d\}) \cap f(\{c, d\})$.[7] Then (PE) implies that $b \in f(\{a, b, c, d\})$ and $d \in f(\{a, b, c, d\})$. Since f is interchangeable, and since $a_1 = b_1$ and $a_2 = d_2$, $a = (b_1, d_2)$ must also be chosen; i.e., $a \in f(\{a, b, c, d\})$. Likewise, $c = (d_1, b_2)$ implies $c \in f(\{a, b, c, d\})$. Finally, (PC) implies that $a \in f(\{a, d\})$, which contradicts $d \succ^{*} a$. So, there cannot be any cycle with length 4.

Now, let us make the induction hypothesis that there is no cycle of length $2(n-1)$ where $n \geq 3$. Given this hypothesis, we prove that there is no cycle of length $2n$.

By reordering the list of individual actions for player 1 and for player 2 from a cycle $a^1 \succeq a^2 \succeq \cdots a^{2n} \succ a^1$, we can generate the table of joint actions in Figure 3. Here, the dashed arrows and the solid arrow represent the links in the cycle as in Figure 2.

The proof by induction argument requires the following steps. Step 1 to 3 gives preferences shown in Figure 8, and Step 4 shows other preferences as reflected in Figure 10-(ii). Step 5 shows the contradiction of these preferences identified in Step 1 to 3 and Step 4.

[7] Again for player 2, $\succeq^{**}$ is defined inversely from the convention of base relation. Accordingly, arrows in the figures inversely represent player 2's revealed preference.

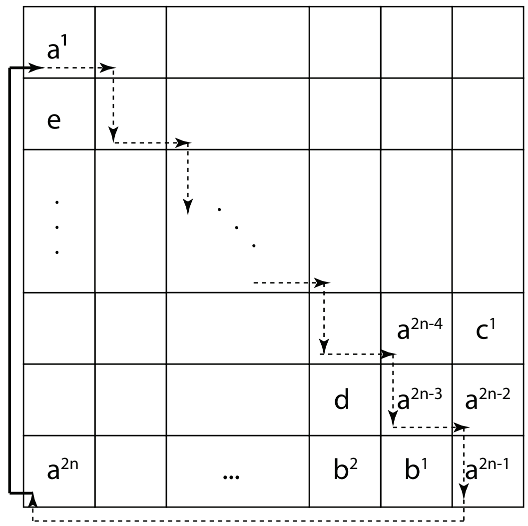

Figure 3: A cycle of length 2n ($n \geq 3$)

Step 1: Consider the feasible set $\{a^{2n-3}, a^{2n-2}, a^{2n-1}, b^1\}$. In addition to the known preferences from the cycle, we can verify $f(\{a^{2n-3}, b^1\})$ and $f(\{b^1, a^{2n-1}\})$. The four cases in Figure 4 below contain all possible cases of $f(\{a^{2n-3}, b^1\})$ and $f(\{b^1, a^{2n-1}\})$. In these two feasible sets, it must not be the case that either $a^{2n-3} \in f(\{a^{2n-3}, b^1\})$ and $a^{2n-1} \in f(\{b^1, a^{2n-1}\})$ (fig (i)), or $b^1 \in f(\{a^{2n-3}, b^1\})$ and $b^1 \in f(\{b^1, a^{2n-1}\})$ (fig (ii)).

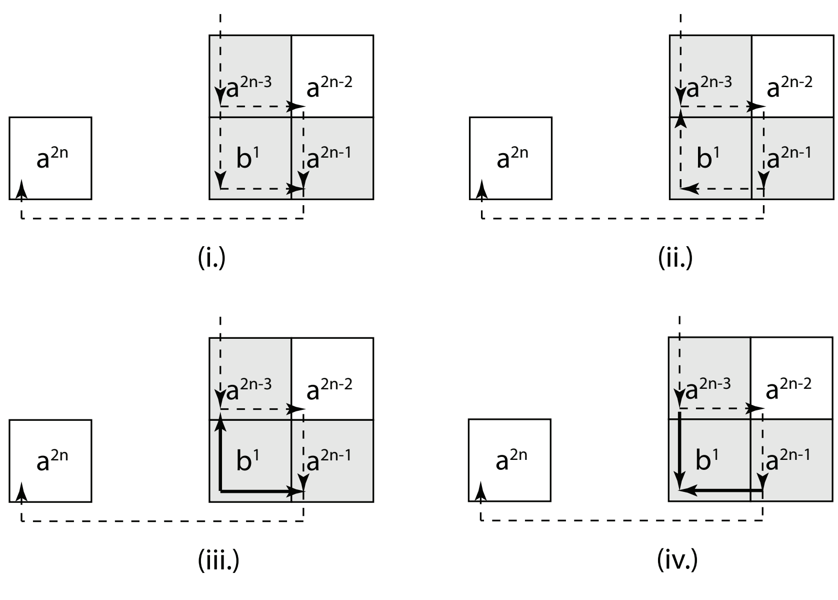

Figure 4: A part of the cycle with length $2n$

In case (i), $a^{2n-4} \succeq^* b^1$ by transitivity of $\succeq^*$ in the left column, and $b^1 \succeq^{**} a^{2n}$ by transitivity of $\succeq^{**}$ in the bottom row. These two preferences induce the cycle $a^1 \succeq \cdots \succeq$

$a^{2n-4} \succeq b^1 \succeq a^{2n} \succ a^1$ which has length $2(n-1)$, a contradiction. In case (ii), $b^1 \in f(\{a^{2n-3}, b^1\}) \cap f(\{b^1, a^{2n-1}\})$ and $a^{2n-2} \in f(\{a^{2n-3}, a^{2n-2}\}) \cap f(\{a^{2n-2}, a^{2n-1}\})$. (PE) induces that a^{2n-2} and b^1 are in $f(\{a^{2n-3}, a^{2n-2}, a^{2n-1}, b^1\})$; interchangeability of f implies that all four joint actions are in $f(\{a^{2n-3}, a^{2n-2}, a^{2n-1}, b^1\})$. Therefore, we have an indifference relation $\sim$ in $\{a^{2n-3}, b^1\}$ and $\{b^1, a^{2n-1}\}$, which gives a special case of (i).

Excluding case (i) and (ii), either (iii) or (iv) must be true. We will prove that the induction step is true in case (iii). The proof in case (iv) is omitted here as it can be shown with exactly the same approach as that taken in case (iii).

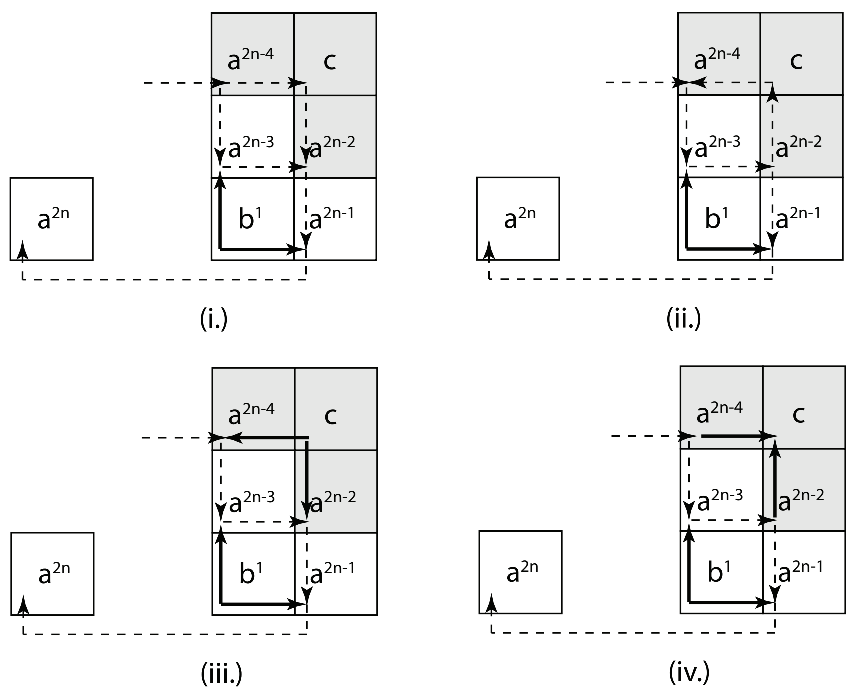

Figure 5: Verifying more preferences

Step 2: Figure 5 contains every possible case of $f(\{a^{2n-4}, c^1\})$ and $f(\{c^1, a^{2n-2}\})$. Using the same argument used for the case (i) and (ii) of $f(\{a^{2n-3}, b^1\})$ and $f(\{b^1, a^{2n-1}\})$ in Step 1, we can rule out the cases of (i) and (ii) in Figure 5. In addition, case (iii), $\{a^{2n-4}\} = f(\{a^{2n-4}, c^1\})$ and $\{c^1\} = f(\{c^1, a^{2n-2}\})$, is not possible either. This can be shown first by observing $b^1 \succ^* a^{2n-4}$. If it is not the case, completeness of $\succeq^*$ in the left column gives $a^{2n-4} \succeq^* b^1$ which, combined with $b^1 \succeq^{**} a^{2n}$ by transitivity of $\succeq^{**}$ in the bottom row, induces the cycle $a^1 \succeq^{**} \cdots \succeq^{**} a^{2n-4} \succeq^* b^1 \succeq^{**} a^{2n} \succ^* a^1$ whose length is $2(n-1)$.

Once (iii) and $b^1 \succ^* a^{2n-4}$ are obtained (see Figure 6), we consider the set of joint actions $\{a^{2n-4}, c^1, b^1, a^{2n-1}\}$. Any choice from this feasible set violates the (PC) in one feasible subset of $\{a^{2n-4}, c^1, b^1, a^{2n-1}\}$. Suppose $c^1 \in f(\{a^{2n-4}, c^1, b^1, a^{2n-1}\})$, then $c \notin f(\{a^{2n-4}, c^1\})$ violates (PC). Likewise any joint action in $\{a^{2n-4}, c^1, b^1, a^{2n-1}\}$ is not a choice. Thus case (iv), $\{c^1\} = f(\{a^{2n-4}, c^1\})$ and $\{a^{2n-2}\} = f(\{c^1, a^{2n-2}\})$, must be true.

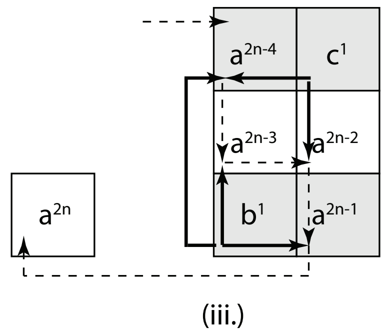

Figure 6: Ruling out the case (iii)

Step 3: Considering $f(\{a^{2n-5}, d\})$ and $f(\{d, a^{2n-3}\})$, we can rule out the cases of either $a^{2n-5} \in f(\{a^{2n-5}, d\})$ and $a^{2n-3} \in f(\{d, a^{2n-3}\})$, or $d \in f(\{a^{2n-5}, d\})$ and $d \in f(\{d, a^{2n-3}\})$ by the same argument used for $f(\{a^{2n-3}, b^1\})$ & $f(\{b^1, a^{2n-1}\})$ and $f(\{a^{2n-4}, c\})$ & $f(\{c, a^{2n-2}\})$ in the previous steps. Accordingly, we only have cases of either $\{a^{2n-5}\} = f(\{a^{2n-5}, d\})$ and $\{d\} = f(\{d, a^{2n-3}\})$, or $\{d\} = f(\{a^{2n-5}, d\})$ and $\{a^{2n-3}\} = f(\{d, a^{2n-3}\})$; case (i) or case (ii) in Figure 7, respectively. Case (i) is ruled out because once we have $a^{2n-5} \succ^* d$, it must be that $a^{2n-2} \succ^{**} d$. If this is not true, then $d \succeq^{**} a^{2n-2}$, which induces one of the following cases.

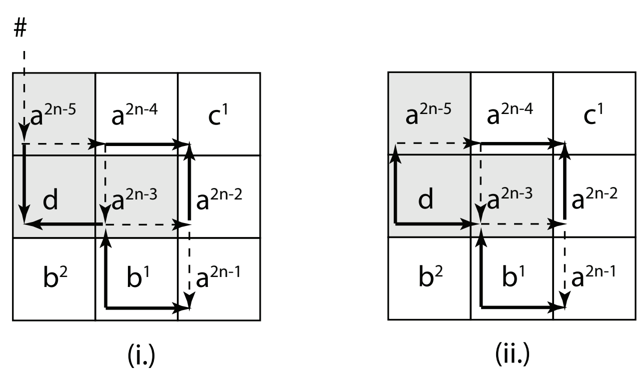

Figure 7: Verifying more preferences.

1. If the cycle has length 6 (a^{2n-5} is a^1 and there is no # in fig(i)), b^2 is equal to a^{2n}. Thus we have $a^{2n-1} \succeq^{**} b^2$ and $b^2 \succ^* d$ by transitivity of $\succeq^*$. As a result, $d \succeq^{**} a^{2n-2}$

makes a cycle with length 4, $d \succeq^{**} a^{2n-2} \succeq^{*} a^{2n-1} \succeq^{**} b^2 \succ^{*} d$, which contradicts the induction hypothesis.

2. If the cycle has length 8 or more (there is a^{2n-6} , '#' in the fig (i), which is not a^1), $a^{2n-6} \succeq^{*} d \succeq^{**} a^{2n-2}$ by transitivity of $\succeq^{*}$ and $\succeq^{**}$ in the left column and the middle row. These preferences shorten the cycle, which contradicts the induction hypothesis.

Therefore, $a^{2n-2} \succ^{**} d$ must be true in case (i). Regardless of what is in $f(\{a^{2n-5}, d, c^1, a^{2n-2}\})$, it violates (PC). For instance, if $d \in f(\{a^{2n-5}, d, c^1, a^{2n-2}\})$ then it must be $d \in f(\{a^{2n-5}, d\})$, which violates $a^{2n-5} \succ^{*} d$. Consequently, case (ii) in Figure 7 must be the option.

By applying Step 2 and 3 sequentially, we can verify more preferences. Figure 8 summarizes the result of this process. In the following proof, Step 4 is necessary only for a cycle whose length is at least 8. For a cycle with length 6, we already know all the preferences that we will verify in Step 4.

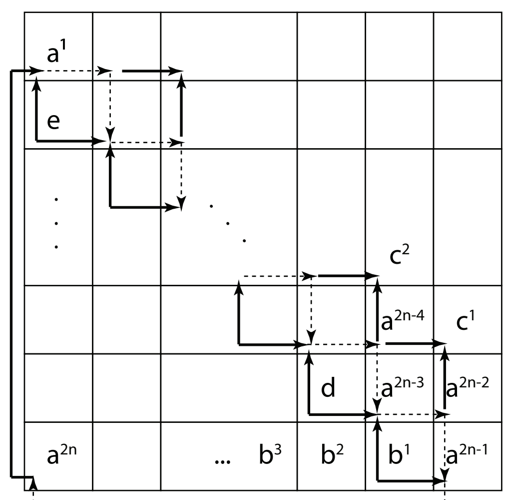

Figure 8: Preferences verified in Step 2 and 3

Step 4: Denote the joint action $(a_1^{2n-1}, a^{2(n-k)-1})$ as b^m and the joint action $(a_1^{2(n-m-1)}, a_2^{2(n-m)})$ as c^m, where $k = 1, 2, \ldots, n-2$. Figure 8 shows where b^m and c^m $(1 \leq m \leq n-2)$ are located. Let τ be a function from $\{b^1, b^2, \ldots, b^{n-2}\}$ to A such that $\tau(b^m) = (a_1^{2n-(2m+1)}, b_2^m)$. Figures 9, 10, and 11 show how the function values are located in the feasible set table. ($\tau(b^m)$ takes its place on the stairway of which b^m is at the bottom.) We prove the following claim.

Claim 10 *For any b^m $(1 \leq m \leq n-2)$, $b^m \succ \tau(b^m)$ and $b^m \succ a^{2n-1}$*

Proof: We prove by induction. Note that we already proved in Step 2 that this claim holds for b^1.

Induction 1: The claim holds for b^2. That is, $b^2 \succ^* \tau(b^2)$ (or a^{2n-5}) and $b^2 \succ^{**} a^{2n-1}$.

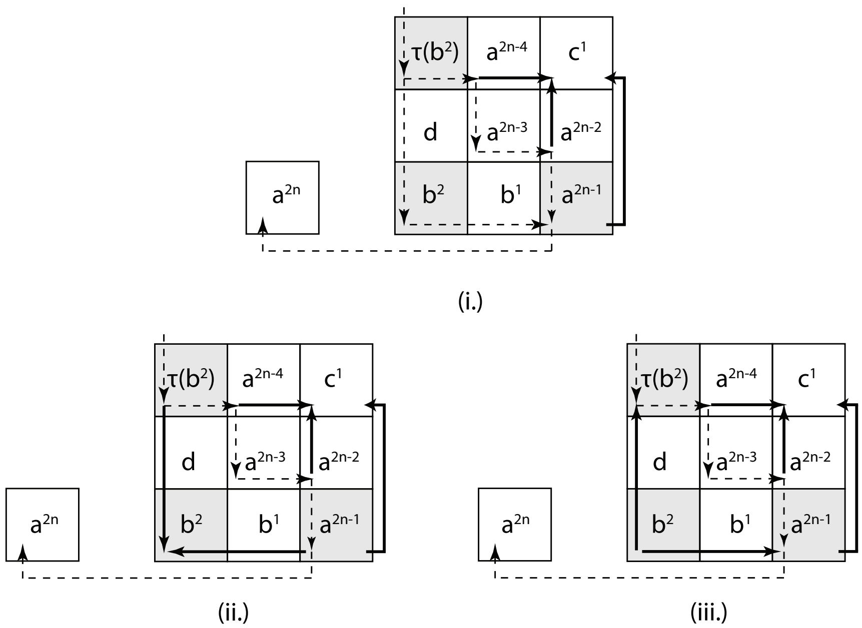

Figure 9: Verifying more preferences involving b^2

Proof: Considering feasible sets, $\{\tau(b^2), b^2\}$ and $\{b^2, a^{2n-1}\}$ (see Figure 9), it is not the case that $\tau(b^2) \in f(\{\tau(b^2), b^2\})$ and $a^{2n-1} \in f(\{b^2, a^{2n-5}\})$ (case (i)). Otherwise, it shortens the cycle with $a^{2n-5} = \tau(b^2) \succeq^* b^2 \succeq^{**} a^{2n}$. (We used transitivity in the bottom row.) Therefore, by completeness in each line, we should have either $a^{2n-1} \succ^{**} b^2$ or $b^2 \succ^* \tau(b^2)$. In the former case, in order not to have a cycle of length 6, which includes $\{\tau(b^2), a^{2n-4}, a^{2n-3}, a^{2n-2}, a^{2n-1}, b^2\}$, f must give $\tau(b^2) \succ^* b^2$ (fig (ii)). In the latter case, in order not to have a cycle of length 6, f must give $b^2 \succ^{**} a^{2n-1}$ (fig (iii)). However, case (ii) is ruled out by considering the feasible set, $\{\tau(b^2), c^1, b^2, a^{2n-1}\}$. To demonstrate this, note that $a^{2n-1} \succ^* c^1$. Otherwise, $\tau(b^2) \succeq^{**} c^1 \succeq^* a^{2n-1}$ shortens the cycle. If case (ii) is true, then any choice from $\{\tau(b^2), c^1, b^2, a^{2n-1}\}$ violates (PC). For example, if $\tau(b^2) \in f(\{\tau(b^2), c^1, b^2, a^{2n-1}\})$, then it must be true that $\tau(b^2) \in f(\{\tau(b^2), c^1\})$. This contradicts $\tau(b^2) \succ^{**} c^1$. (Note again that $\succeq^{**}$ is defined inversely.) Therefore, (iii) must be the case in Figure 9. ∎

Induction 2: If the claim holds for b^{m-2}, it also holds for b^m $(3 \leq m \leq n-2)$.

Proof: With the same approach as Induction 1, f should not give $\tau(b^m) \succeq^* b^m$ and $b^m \succeq^{**} a^{2n-1}$; otherwise, we have a shorter cycle including $\tau(b^m) \succeq^* b^m \succeq^{**} a^{2n}$. Thus, it must be either $a^{2n-1} \succ^{**} b^m$ or $b^m \succ^* \tau(b^m)$. In the former case, not to have a cycle, $b^m \succeq^* \tau(b^m) \succeq^{**} \cdots \succeq^* a^{2n-1} \succ^{**} b^m$ which has length $2m+2 \leq 2(n-1)$, it must be true that $\tau(b^m) \succ^* b^m$ (case (i) in Figure 10).[8] In the latter case, not to have a cycle, $\tau(b^m) \succeq^{**} \cdots \succeq^* a^{2n-1} \succeq^* b^m \succ^* \tau(b^m)$ which has length $2m+2 \leq 2(n-1)$, it must be true that $b^m \succ^{**} a^{2n-1}$. (case (ii) in Figure 10.) However, case (i) is ruled out. First, observe that $b^{m-2} \succ^* c^{m-1}$ must be true; otherwise $\tau(b^m) \succeq^{**} c^{m-1} \succeq^* b^{m-2} \succeq^{**} a^{2n}$ leads to a shorter cycle. In addition, transitivity of $\succeq^{**}$ in the bottom row gives $b^{m-2} \succ^{**} b^m$. Then, in the feasible set, $\{\tau(b^m), b^m, b^{m-2}, c^{m-1}\}$, any choice violates (PC). Therefore, (ii) must be the case in $f(\{\tau(b^m), b^m\})$ and $f(\{b^m, a^{2n-1}\})$. ∎

By induction, $b^m \succ \tau(b^m)$ and $b^m \succ a^{2n-1}$ for $m = 1, \ldots, n-2$. Claim 10 holds. ∎

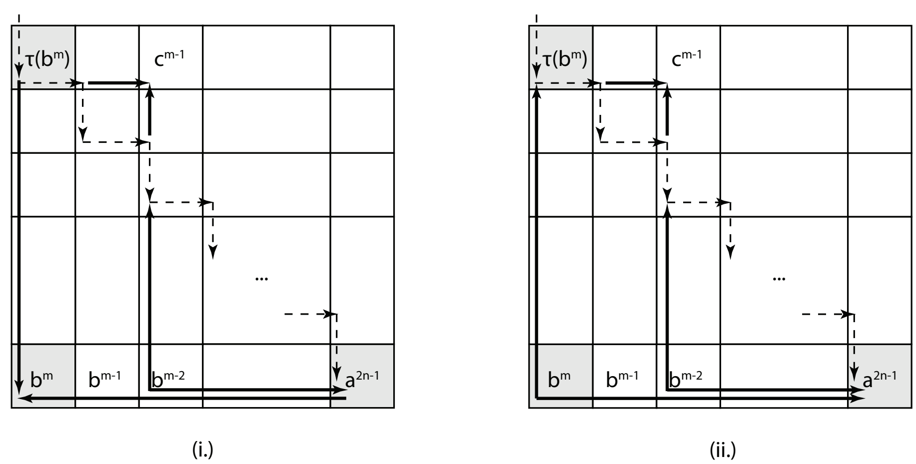

Figure 10: Verifying preferences involving b^m

Step 5: Results from Steps 2 and 3, and results from Step 4 contradict each other.

Proof: If we denote the joint action $(\tau(b^{n-2})_1, a_2^1)$ as e (see Figure 11), then Step 2 and 3 gives $e \succ^* a^1$ and $e \succ^{**} \tau(b^{n-2})$. We showed in Step 4 that $b^{n-2} \succ^* \tau(b^{n-2})$ and $b^{n-2} \succ^{**} a^{2n-1}$. Moreover, it must be true that $e \succ^* a^{2n}$, since otherwise, $a^{2n} \succeq^* e \succ^{**} a^4$ shortens the cycle.

[8] Although we explicitly write the proof only for the case of cycle beginning with $\succeq^{**}$, every single step so far could have been reproduced for cases where cycles begin with $\succ^*$. Here, we used the induction hypothesis, "there is no cycle with a length of $2(n-1)$," from the counterpart proof of cycles begining with $\succeq^*$.

On the other hand, $b^{n-2} \succ^{**} a^{2n}$ by transitivity of $\succeq^{**}$ in the bottom row. We can observe that any choice from the feasible set, $\{e, \tau(b^{n-2}), a^{2n}, b^{n-2}\}$, violates (PC). This contradiction completes the proof of Step 5, thereby completing the proof of consistency of $\succeq$. ∎

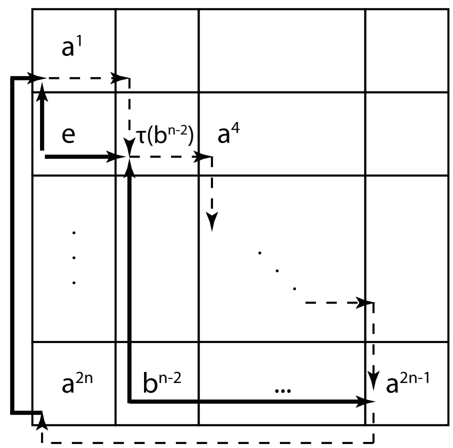

Figure 11: A contradiction

A.2 Characterizing a rationalizing preference relation.

Claim 11 *For all $B \in \mathcal{A}$, $f(B)$ coincides with the set of all Nash equilibria of the game $(B, \succeq, \preceq)$.*

Proof: Take any $B = B_1 \times B_2 \in \mathcal{A}$, and let NE$(B)$ be the set of all Nash equilibria of the game $(B, \succeq, \preceq)$. First, to show $f(B) \subseteq \text{NE}(B)$, we take any $b^* = (b_1^*, b_2^*) \in f(B)$. Since f satisfies (PC), $b^* \in f(B')$ for all $B' \in \mathcal{A}$ and $B' \subseteq B$. Therefore, for any $\{b^*, (b_1, b_2^*)\} \subseteq B$, $b^* \in f(\{b^*, (b_1, b_2^*)\})$. By the definition of $\succeq^*$, we have $b^* \succeq^* (b_1, b_2^*)$, which is equal to $b^* \succeq (b_1, b_2^*)$. Similarly, for any $\{b^*, (b_1^*, b_2)\} \subseteq B$, $b^* \in f(\{b^*, (b_1^*, b_2)\})$. The definition of $\succeq^{**}$ gives $(b_1^*, b_2) \succeq^{**} b^*$, which is equal to $(b_1^*, b_2) \succeq b^*$, or $b^* \preceq (b_1^*, b_2)$. Since $b^* \succeq (b_1, b_2^*)$ and $b^* \preceq (b_1^*, b_2)$, for all $(b_1, b_2^*) \in B$ and $(b_1^*, b_2) \in B$, b^* is a Nash equilibrium of the game $(B, \succeq, \preceq)$.

Conversely, if $b^* \in \text{NE}(B)$, for any $(b_1, b_2^*) \in B$, $b^* \succeq (b_1, b_2^*)$. Since, only $\succeq^*$, and not $\succeq^{**}$, is defined in columns, we have $b^* \succeq^* (b_1, b_2^*)$. The definition of $\succeq^*$ gives $b^* \in f(\{b^*, (b_1, b_2^*)\})$, and (PE) implies $b^* \in f(B_1 \times \{b_2^*\})$ (#). $b^* \in \text{NE}(B)$ implies $b^* \preceq (b_1^*, b_2)$ for all $(b_1^*, b_2) \in B$ (or $(b_1^*, b_2) \succeq b^*$). Because we defined only $\succeq^{**}$, and not $\succeq^*$, in rows, we have $(b_1^*, b_2) \succeq^{**} b^*$. The definition of $\succeq^{**}$ gives $b^* \in f(\{b^*, (b_1^*, b_2)\})$ and (PE) induces $b^* \in f(\{b_1^*\} \times B_2)$ (##). Lastly, (#), (##), and (PE) imply that $b^* \in f(B)$. ∎

We have shown that $\succeq$ is consistent and $f(B)$ coincides with NE(B) for all $B \in \mathcal{A}$. Suzumura (1976) shows that a consistent relation has a weak order extension. Since the extension generates additional preferences only between two joint choices which are not in a line, this extension does not affect the result of Claim 11. Therefore, Claim 11 is still valid with the weak order extension of $\succeq$. This completes the proof of the main theorem.

References

Austen-Smith, D., and J. S. Banks (1994): *Positive Political Theory I : Collective Preference*. Univ. of Michigan Press, Ann Arbor.

Carvajal, A., R. Deb, J. Fenske, and J. Quah (2010): "Revealed Preference Tests of the Cournot Model," *Working Paper No. 14998, University of Oxford.*

Chambers, C., and F. Echenique (2011): "Testable implications of Bargaining Theories," *Social Science Working Paper No. 1348, Caltech.*

Cherchye, L., T. Demuynck, and B. De Rock (2011): "The Empirical Content of Cournot Competition," *Working Paper, Erasmus Research Institute of Management.*

Demuynck, T., and L. Lauwers (2009): "Nash rationalization of collective choice over lotteries," *Mathematical Social Sciences*, 57(1), 1–15.

Duersch, P., J. Oechssler, and B. Schipper (2011): "Unbeatable Imitation," *Working Paper, University of California, Davis.*

——— (forthcoming): "Pure Strategy Equilibria in Symmetric Two-Player Zero-Sum Games," *International Journal of Game Theory.*

Echenique, F., and L. Ivanov (2011): "Implications of Pareto efficiency for two-agent (household) choice," *Journal of Mathematical Economics*, 47(2), 129–136.

Forges, F., and E. Minelli (2009): "Afriat's theorem for general budget sets," *Journal of Economic Theory*, 144(1), 135–145.

Galambos, A. (2009): "The complexity of nash rationalizability," *Working Paper, Lawrence University.*

Luce, R., and H. Raiffa (1989): *Games and decisions: Introduction and critical survey.* Dover Pubns.

Moulin, H. (1985): "Choice functions over a finite set: A summary," *Social Choice and Welfare*, 2(2), 147–160.

Peleg, B., and S. Tijs (1996): "The consistency principle for games in strategic form," *International Journal of Game Theory*, 25(1), 13–34.

Plott, C. (1974): "On game solution and revealed preference theory," *Social Science Working Paper No. 35, Caltech.*

Radzik, T. (1991): "Saddle point theorems," *International Journal of Game Theory*, 20, 23–32.

Ray, I., and S. K. Snyder (2003): "Observable Implications of Nash and Subgame - Perfect Behavior in Extensive Games," *Working Paper, University of Birmingham.*

Ray, I., and L. Zhou (2001): "Game Theory via Revealed Preferences," *Games and Economic Behavior*, 37(2), 415 – 424.

Richter, M. (1971): "Rational choice," in *Preferences, Utility, and Demand*, pp. 29–58. Harcourt Brace Jovanovich, New York.

Samuelson, P. A. (1938): "The Empirical Implications of Utility Analysis," *Econometrica*, 6(4), 344–356.

Sen, A. K. (1971): "Choice Functions and Revealed Preference," *Review of Economic Studies*, 38(115), 307–17.

Shapley, L. (1964): "Some Topics in Two Person Games," in *Advances in Game Theory*, ed. by M. Dresher, L. Shapley, and A. Tucker, pp. 1–28. Princeton University Press.

Sprumont, Y. (2000): "On the Testable Implications of Collective Choice Theories," *Journal of Economic Theory*, 93(2), 205 – 232.

Suzumura, K. (1976): "Remarks on the Theory of Collective Choice," *Economica*, 43(172), 381–390.

Wilson, R. (1970): "The finer structure of revealed preference," *Journal of Economic Theory*, 2(4), 348–353.

Xu, Y., and L. Zhou (2007): "Rationalizability of choice functions by game trees," *Journal of Economic theory*, 134(1), 548–556.