Mergers and Acquisitions with Private Equity Intermediation*

Swaminathan Balasubramaniam†, Armando Gomes‡, SangMok Lee§

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Abstract

We develop a search model of mergers and acquisitions (M&A), intermediated by private equity (PE) funds which may face pressure to sell. The selling pressure leads to the development of a secondary buyout (SBO) market, enabling PE funds bail each other out. Interestingly, an increase in the number of PE funds can improve each fund's value, because the enhanced benefits of SBOs can prevail over the reduction in value from narrower buy-sell spreads due to more intense competition. We calibrate the model using data for the US middle market and find that PE funds could lose 64% of their valuation without SBOs. Moreover, the increase in the number of funds from 2000 to 2017 contributes to a 48% increase in fund valuation due to the complementarity among funds. Nevertheless, our model predicts that this mechanism might have peaked in 2021, and more PE funds could decrease their value.

Keywords: Mergers and Acquisitions, Private Equity, Secondary Buyouts, OTC Markets, Financial Intermediation

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†NEOMA Business School, `s.balasubramaniam@neoma-bs.fr`

‡Olin Business School, Washington University in St. Louis, `gomes@wustl.edu`

§Department of Economics, Washington University in St. Louis, `sangmoklee@wustl.edu`

1 Introduction

Financial intermediaries that purchase assets and retain them until resale often rely on external capital. This external funding exposes them to the risk of selling assets under pressure. One example of such intermediaries is closed-ended private equity (PE) buyout funds that operate within the corporate acquisition market. A PE fund acquires a small number of assets such as corporate subdivisions, and exits primarily by selling the assets to a strategic buyer in a related industry, or secondary buyout (SBO) which entails selling to another PE fund. PE buyout funds have become significant players in the corporate acquisition market. In 2017, their investments accounted for \$538 billion out of the total \$2.1 trillion in the US mergers and acquisition (M&A) market, representing 4,053 out of 10,769 deals, and over the period from 2011 to 2017, these funds experienced a compounded annual growth rate of 7.5%.[1]. Typically, a buyout fund operates for 10-12 years, and as it nears the end of its life span, it may sell assets under pressure, often to other PE funds in the secondary market (Arcot et al., 2015).[2]

We build a search-theoretic model of asset reallocation, where intermediaries (hereafter, *funds*) are at such risk of selling under pressure, considering PE buyout funds in the M&A market as a key application. The model introduces (corporate) investors who can buy and sell assets (firms or corporate divisions) directly, and (PE) funds that intermediate between investors. The search-and-bargaining features are suitable for capturing intermediaries' attempts to sell in a decentralized market. In particular, PE funds often take several months to find an appropriate buyer and close a transaction. We explore the impact of market characteristics, such as search friction and the number of funds on fund valuations, trading volumes, transaction prices, and welfare.

Our model comprises a continuum of investors and funds who hold either one or zero

[1] The data on PE activities in the M&A market is from PitchBook Inc, and can be found at `https://pitchbook.com/news/reports/2q-2018-ma-report`

[2] A PE fund consists of General Partners (GPs) who have specialties in specific industries and operating expertise such as finance and marketing. The GPs raise capital from outside investors, called Limited Partners (LPs). After raising capital, a typical PE fund usually has around 10 to 12 years of life after its inception. The rationale for a finite life is that funds invest in private firms whose market value is unknown. Only after an asset is sold, GPs and LPs observe gains of the fund and can determine GPs' management compensation and LPs' share.

assets. The assets are reallocated over time through investor-to-investor direct trading, fund-investor trading, and fund-to-fund secondary trading with heterogeneous search frictions. High-type investors can generate higher flow payoffs, so assets are reallocated from low-type investors to high-type investors, sometimes through transactions intermediated by funds. While holding assets, funds may receive liquidity shocks and incur a holding cost from owing assets. Then, the funds under selling pressure attempt to sell their assets either to investors or to other funds through secondary transactions.

We calibrate our model to the US mid-size corporate acquisition market with intermediation by PE middle-market (MM) funds to quantitatively address equilibrium questions involving complex interactions among parameters. The mid-size corporate acquisition market lends itself naturally to a search framework: the middle-market is particularly unintegrated with approximately 100,000 corporations and almost 2,000 PE MM funds, and it takes about a year for sellers to find appropriate buyers and close transactions. A typical transaction involves a sale of all the corporation assets or a subdivision by corporate investors or PE buyout funds.

Our equilibrium analysis centers on the (fund-to-fund) secondary market. First, a faster search in the secondary market enables funds to escape liquidity constraints more easily, mitigating the reduction of the fund's value upon receiving a liquidity shock. Moreover, since value-generating assets can be more easily transferred from funds under selling pressure to other funds, a faster search in the secondary market improves overall welfare (Proposition 3). Second, funds sourcing deals provide liquidity to funds in the exit phase, while funds at the exit phase enable fund buyers to acquire assets more quickly. A faster secondary market improves this channel through which funds can complement each other (Proposition 4). As a result, perhaps surprisingly, the value of funds can increase with the number of funds (Proposition 5).

The number of PE MM funds has significantly increased from 293 to 1,994 from 2000 to 2017 (see Table 1). Our calibration, based on 2012 data, shows that this rise in fund numbers, while holding all else constant, contributes to a 48.7% increase in fund valuation. However, our model predicts that this trend – more PE funds increase each one's valuation – would reach its saturation point when the number of funds reaches 2,375. Further increases in fund numbers beyond this threshold can exert pressure on fund valuations due to heightened competition among funds. Interestingly, the number of PE funds declined from the peak levels of 2,472 funds in 2021 to 1,838 funds in 2023, aligning with the saturation point predicted by our calibration.

Secondary transactions are sometimes criticized as opportunistic behavior among fund managers passing sub-par assets to their counterparts, all the while both sides collect management fees from the fund investors (Arcot et al. (2015)).[3] Concerns also arise due to the limited potential for operational value creation in SBOs, as initial private equity sponsors may have already exploited the most accessible value-creation measures (Bonini (2015), Wang (2012)). Accordingly, our model conservatively assumes that a fund's expected flow payoffs under SBOs is the same as that under a primary buyout (PBO), thus there is no operating value improvement. Yet, we find that funds generate higher returns because of (and not in spite of) the possibility of secondary transactions. SBOs provide firms with an avenue to smoothly transition ownership towards the conclusion of a fund's lifecycle (rather than focusing solely on potential for operational improvement). A counterfactual exercise of shutting down the secondary market shows that PE fund values could be 64% lower if SBOs are disallowed. This observation is aligned with Hammer et al. (2022) and Harford and Kolasinski (2014) who emphasize the role of longer fund portfolio duration in firm value creation.[4]

A further quantitative result is that our model calibration approximates the percentages of SBO exits from 2007 to 2017 associated with the increase in the number of PE funds. While it is evident that the number of secondary trades increases with the growth in funds, our model explains a tandem increase in the *share* of exits through secondary trades.

We also study various equilibrium properties, including welfare, trade speeds, prices, and trade volumes. Notably, we observe that direct trades between low and high-type investors may diminish overall welfare, as investors' direct trading deprives funds of potential trade opportunities. Specifically, funds will find it harder to turn over their inventory quickly when they are under selling pressure. Restricting direct trading for investors can facilitate exit-phase funds in promptly offloading assets, particularly when the funds can identify

[3] See also an article in *The Economist* at `https://www.economist.com/node/15580148`.

[4] Hammer et al. (2022) analyze private equity-backed buy-and-build (B&B) strategies, which often rely on multiple add-on acquisitions. They emphasize that synergies among portfolio companies may take a long time to form, hence leading to fund exits through secondary buyouts.

trading opportunities fast. This kind of welfare benefit of secondary transactions has also been studied in a banking context - Pagano and Volpin (2012) show that high securitization activities in the secondary market yield high loan issuance in the primary market.

Broadly, our paper belongs to the literature that builds search theory models to study the M&A market. To list a few, Jovanovic and Rousseau (2002)'s "q-theory of mergers" suggests that high-type corporations (i.e., high market-to-book ratio, q) often acquire low-type corporations to create value, whereas Rhodes-Kropf et al. (2005) and Rhodes-Kropf and Robinson (2008) find evidence of like-buys-like and explain such assortative matching with buyer-seller complementarities under low search frictions.[5] It is worth noting that we are agnostic to the explanations of M&As. We take M&A activities as given and instead focus on PE intermediation and their secondary market.

Our paper is the first to study PE intermediation in the M&A market through a search-theoretic model, in the spirit of the Over-the-Counter (OTC) market literature (Duffie et al. (2005)).[6] Hugonnier et al. (2020) examine an inter-dealer market, sharing some similarities with our direct and secondary trading markets. However, key distinctions exist. Our model is specifically designed to capture funds facing selling pressure from liquidity shocks, which arise based on asset holdings and dissipate when funds off-load assets. This is particularly relevant for funds intermediating with external capital. Additionally, while their inter-dealer market is singled out such that investors trade only through dealers, we allow for simultaneous interactions among investors and funds. This unrestricted interaction is crucial for applications like the corporate acquisition and real estate markets (Phillips and Zhdanov (2017)). Lastly, while their results focus on trading patterns and intermediation chains, our emphasis is on fund valuations and welfare. Due to their moderate flow payoffs, funds in our model self-select to intermediate between low and high-type investors, similar to studies in the recent OTC literature Neklyudov (2019), Üslü (2019), Nosal et al. (2016), Shen et al. (2021), Yang and Zeng (2018), and Farboodi et al. (2017).[7]

[5] Other papers relying on q-theory include Eisfeldt and Rampini (2006) and Eisfeldt and Rampini (2008) which study capital reallocations. Also, David (2017) develops a search and matching M&A model to evaluate the implications of merger activity for aggregate economic outcomes.

[6] We exclusively address closely related studies, directing readers to Nosal and Rocheteau (2011) for additional references. Notably, our model contrasts with interbank network models examined by Allen and Babus (2009), which primarily center on lending and borrowing.

[7] It is often the case that mid-type investors, akin to our funds, choose to intermediate with a comparative

The remainder of the paper is organized as follows: Section 2 introduces the formal model; Section 3 provides equilibrium properties; Section 4 presents a calibration exercise; Section 5 discusses our main analysis of secondary transactions; Section 6 provides results on welfare, prices, and trade volumes; and Section 7 concludes.

2 Model

Time runs continuously in $t \in [0, \infty)$. Over time, two kinds of agents, **investors** and **funds**, trade **assets**. In the context of the corporate acquisition market, these agents represent corporate investors and PE funds that trade firms or corporate divisions. Initially, a fraction of investors and funds are endowed with assets. The measures of investors k_v, funds k_f, and tradable assets k_a remain constant. All agents are risk neutral and infinitely lived, with time preferences determined by a constant discount rate r. Each agent holds one or zero assets.[8] Hence, $k_a < k_v + k_f$. We normalize the total measure of investors as $k_v = 1$.

An investor that holds an asset generates either a high payoff flow u_h or a low payoff flow u_l ($< u_h$). An investor does not receive any payoff flow when not holding an asset. An investor's ability to create payoff flow switches from low to high with Poisson intensity ρ_u, or from high to low with intensity ρ_d. The arrival rate of this Poisson shock for each type of investor is independent of other investors.[9] The set of investor types is $\mathcal{T}_v \equiv \{ho, lo, hn, ln\}$, where the letters h and l represent each investor's ability to generate payoffs and the letters o and n denote whether an investor owns an asset or not.

A fund's life cycle consists of an investment phase, a harvesting phase, and an exit phase.

advantage in search skills. In the case of derivative swap contracts, investors with risky endowments may face size limits on bilateral trades, preventing full risk sharing. This leads to price dispersion, motivating some banks to act as intermediaries in the OTC market, as observed by Atkeson et al. (2015).

[8]Certainly, our model contains some simplifications. Private equity funds in practice can hold multiple assets and directly improve the fundamental value of the holding assets.

[9]Although beyond the scope of the current paper, our model can be adapted to accommodate systematic shocks to the economy by introducing state variables and transitions from one state to another. Using economic state variables to capture aggregate shocks appears in the M&A literature. For example, acquisitions motivated by production synergies occur more during economic expansions, whereas disinvestment-related takeovers occur more during recessions (Mason and Weeds (2010); Lambrecht (2004); Lambrecht and Myers (2007). Further, Bernile et al. (2012) show that horizontal mergers in oligopolistic industries tend to occur when consumer demand increases or decreases, rather than remaining stable.

A fund in the investment phase does not own assets and searches for an investor or a fund selling assets. After purchasing an asset, the fund enters the harvesting phase and creates payoff flow u_f. A fund in the harvesting phase sells its assets and starts a new life cycle (i.e., goes back to the investment phase),[10] or it receives a liquidity shock with intensity ρ_e and enters the exit phase. A fund in the exit phase incurs a holding cost and generates a lower payoff flow u_e ($< u_f$). After selling the asset, the fund automatically starts a new life in the investment phase. We denote a fund in the investing phase by type fn (a fund non-owner), in the harvesting phase by type fo (a fund owner), and in the exiting phase by type fe (a fund that is exiting). The set of fund types is $\mathcal{T}_f \equiv \{fn, fo, fe\}$.

We assume that funds generate moderate payoff flows, $u_l < u_e < u_f < u_h$, such that funds play the role of intermediaries by purchasing assets from low-type investors and selling them to high-type investors. In reality, PE buyout funds contribute to operational efficiencies of assets (thus, $u_f > u_l$). However, assets divested by buyout funds are acquired by corporate investors, indicating that the flow payoff for certain corporate investors must be even higher than that of buyout funds (thus, $u_h > u_f$). This disparity may be attributed to additional benefits associated with holding an asset, such as synergies with the investor's existing asset portfolio.

Let $\mathcal{T} \equiv \mathcal{T}_v \cup \mathcal{T}_f$ denote the set of types with typical elements i, j, etc. The measure of type $i \in \mathcal{T}$ at time $t \in [0, \infty)$ is denoted by $\mu_i(t)$. Then,

$$\begin{aligned} \mu_{ho}(t) + \mu_{hn}(t) + \mu_{lo}(t) + \mu_{ln}(t) &= k_v (= 1), \\ \mu_{fn}(t) + \mu_{fo}(t) + \mu_{fe}(t) &= k_f, \\ \mu_{ho}(t) + \mu_{lo}(t) + \mu_{fo}(t) + \mu_{fe}(t) &= k_a. \end{aligned} \tag{1}$$

Agents meet each other over time and negotiate a trade. Two investors meet each other with intensity λ_d for (investor-to-investor) **direct trading**. An investor and a fund meet each other with intensity λ_f for a **fund-investor trading**. A fund in the exit phase (fe) and a fund in the investment phase (fn) meet each other with intensity λ_s for (fund-to-fund) **secondary trading**. The meeting rate between any pair of groups is linear in each group's population. That is, for any pair of investor types $i, j \in \mathcal{T}_v$ with measures μ_i and μ_j, the total

[10] General partners of PE funds often start a new fund around the liquidation of an existing fund.

meeting rate is $\lambda_d \mu_i \mu_j$. Similarly, the total meeting rate between an investor type $i \in \mathcal{T}_v$ and a fund type $j \in \mathcal{T}_f$ is $\lambda_f \mu_i \mu_j$, and the total meeting rate in the secondary market is $\lambda_s \mu_{fe} \mu_{fn}$. When two agents meet each other, they trade an asset instantaneously if and only if the gain from trade (which we explain later) is positive. The assumption of immediate trading upon meeting follows the literature on bargaining without asymmetric information.

We will find an equilibrium in which all tradings denoted by $\mathcal{M} \equiv \{lo\text{-}hn, lo\text{-}fn, fo\text{-}hn, fe\text{-}hn, fe\text{-}fn\}$ are active.[11] That is, an hn-type investor acquires an asset from a lo-type investor (lo-hn trade). An hn-type investor can also acquire an asset from a fund of type either fo or fe (either fo-hn or fe-hn trade). Similarly, an fn-type fund acquires an asset through primary buyout (PBO) from a lo-type investor (lo-fn trade), or through secondary buyout (SBO) from a fe-type fund. After all trades, the types change from 'o' to 'n' and vice versa. Overall, assets are transferred from low-type investors toward high-type investors, with a possible chain of trades among funds through secondary trades.

Figure 1 summarizes the model. Agent types are listed on the left column for owners and the right column for non-owners. An owner changes her type to one on the right column upon selling her asset; non-owner changes her type to one on the left column upon purchasing an asset (a fund's type becomes fo after an asset purchase). The solid arrows represent asset reallocations from sellers to buyers. The vertical dashed arrows represent the exogenous type changes: high vs. low for investors, or a liquidity shock to fo-type funds.

An asset market with fund intermediation is a collection of exogenous parameters $\theta \equiv (k, r, u, \rho, \lambda)$, where $k \equiv (k_v, k_f, k_a)$, $u \equiv (u_l, u_h, u_f, u_e)$, and $\lambda \equiv (\lambda_d, \lambda_f, \lambda_s)$. All exogenously given parameters are strictly positive. We provide a summary of the model parameters in Table 6 in the Appendix, along with the equilibrium variables that will be introduced in the subsequent section.

[11] However, two buyout funds, fo and fn, do not trade. Such circumstance does not yield any gains because both funds would receive the same payoff flow and have an equal chance (Poisson arrival) of experiencing a liquidity shock while holding the asset. For the same reason, the same type of investors, either high-type or low-type, also do not trade. In the real world, a new fund that acquires an asset typically does not dispose of it to another fund before reaching the end of its life cycle. This lack of trading is likely due to insufficient gains from such trades. Our model ensures that the same property holds in equilibrium.

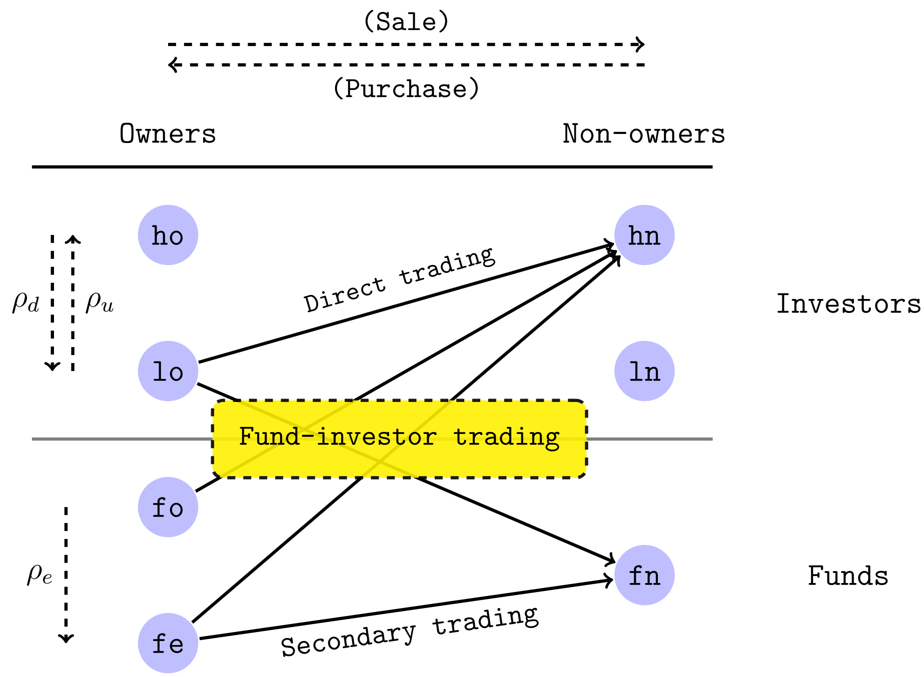

Figure 1: An asset market with fund intermediation. In the application to the M&A market with PE intermediation, direct trading corresponds to the M&A market, where a lo-type corporation sells an asset to a hn-type corporation. Fund-investor tradings correspond to PBO transactions, where a lo-type corporation sells an asset to a fn-type fund ($lo - fn$ trade), or a PE fund of type either fo or fe sells an asset to a hn-type corporation (either $fo - hn$ or $fe - hn$ trade). Secondary tradings (the SBO market) occur when a fe-type fund sells to a fn-type fund ($fe - fn$ trade).

3 Equilibrium

We characterize a steady-state equilibrium in which investors trade assets amongst themselves, and funds intermediate by buying and selling assets. We will calibrate the model by aligning the equilibrium characterization with the data from PE funds in the corporate acquisition market.

We first derive steady-state population measures. Let k_h and k_l denote the steady-state populations of high- and low-type investors. Since we normalized the total measure of investors as $k_v = 1$, from the rates of exogenous type changes,

$$k_h = \frac{\rho_u}{\rho_u + \rho_d} \quad \text{and} \quad k_l = \frac{\rho_d}{\rho_u + \rho_d}.$$

A hn-type investor switches its type to ho upon purchasing an asset from either a lo-type investor or a fo- or fe-type fund. As such, hn-type investors become ho-type at the rate of $(\lambda_v \mu_{lo} + \lambda_f \mu_{fo} + \lambda_f \mu_{fe})\mu_{hn}$. On the other hand, as a result of exogenous type changes, hn-type investors switch their types to ln at the rate of $\rho_d \mu_{hn}$; similarly, ln-type investors switch their types to hn at the rate of $\rho_u \mu_{ln}$. Thus,

$$\dot{\mu}_{hn}(t) = -(\lambda_d \mu_{lo}(t) + \lambda_f \mu_{fo}(t) + \lambda_f \mu_{fe}(t))\mu_{hn}(t) - \rho_d \mu_{hn}(t) + \rho_u \mu_{ln}(t). \tag{μ-hn}$$

The population measures for other types change over time by similar processes:

$$\dot{\mu}_{ho}(t) = (\lambda_d \mu_{lo}(t) + \lambda_f \mu_{fo}(t) + \lambda_f \mu_{fe}(t))\mu_{hn}(t) - \rho_d \mu_{ho}(t) + \rho_u \mu_{lo}(t), \tag{μ-ho}$$

$$\dot{\mu}_{ln}(t) = (\lambda_d \mu_{hn}(t) + \lambda_f \mu_{fn}(t))\mu_{lo}(t) - \rho_u \mu_{ln}(t) + \rho_d \mu_{hn}(t), \tag{μ-ln}$$

$$\dot{\mu}_{lo}(t) = -(\lambda_d \mu_{hn}(t) + \lambda_f \mu_{fn}(t))\mu_{lo}(t) - \rho_u \mu_{lo}(t) + \rho_d \mu_{ho}(t), \tag{μ-lo}$$

$$\dot{\mu}_{fn}(t) = \lambda_f \left(\mu_{hn}(t)\mu_{fo}(t) + \mu_{hn}(t)\mu_{fe}(t) - \mu_{lo}(t)\mu_{fn}(t)\right), \tag{μ-fn}$$

$$\dot{\mu}_{fo}(t) = (\lambda_f \mu_{lo}(t) + \lambda_s \mu_{fe}(t))\mu_{fn}(t) - \lambda_f \mu_{hn}(t)\mu_{fo}(t) - \rho_e \mu_{fo}(t), \tag{μ-fo}$$

$$\dot{\mu}_{fe}(t) = -(\lambda_f \mu_{hn}(t) + \lambda_s \mu_{fn}(t))\mu_{fe}(t) + \rho_e \mu_{fo}(t). \tag{μ-fe}$$

Let $P(\theta)$ denote the above system of population equations (μ-hn)-(μ-fe). A real-vector $\mu \equiv (\mu_i)_{i \in \mathcal{T}}$ with each $\mu_i \geq 0$ is a steady-state solution of $P(\theta)$ if the right-hand sides of the equations, with $\mu_i(t)$ replaced by μ_i for each $i \in \mathcal{T}$, are equal to zero.

Proposition 1. *(Steady-state Population Measures)*

1. *(Existence and Uniqueness) There exists a unique steady-state solution μ of $P(\theta)$ such that $\mu_i > 0$ for all $i \in \mathcal{T}$.*

2. *(Asymptotic Stability) Let $\mu(t)$ be a dynamic solution of $P(\theta)$ with initial condition $\mu(0)$. For any $\epsilon > 0$, there exists $\delta > 0$ such that, if $\|\mu(0) - \mu\| < \delta$, then $\|\mu(t) - \mu\| \leq \epsilon$ for all t, and $\mu(t) \to \mu$ as $t \to \infty$.*

The proof uses the Poincare-Hopf index theorem (Simsek et al., 2007), which generalizes the Intermediate Value Theorem. To get an intuition, we set $(\mu_i)_{i\in\mathcal{T}_v} \approx 0$, while satisfying the population equations $P(\theta)$ but violating $k_v = 1$. A small increase of μ_{lo} (or μ_{hn}) increases the supply (resp., demand) of assets for investors' direct trading $\lambda_d\mu_{lo}\mu_{hn}$, and in turn the number of investors that are rightly holding or not-holding assets (μ_{ho} and μ_{ln}). The increased populations of μ_{ho} and μ_{ln} lead to more inflow $\rho_d\mu_{ho}$ of agents back to the aggregate supply and the inflow $\rho_u\mu_{ln}$ to the aggregate demand for direct trading. That is, all four investor-type populations increase. Taking into account how investor-type populations are related to fund-type populations, we find a unique supply μ_{lo} and demand μ_{hn} that yield $\sum_{i\in\mathcal{T}_v}\mu_i = k_v = 1$ by the index theorem. The second part of the proposition on stability is due to a classical result in dynamical systems.[12]

We define a steady-state equilibrium via a recursive equation of certain values (or, expected utilities). The sources of value to all agents in our model are two-fold: flow payoffs while holding assets and gains from trade. Let v_{hn} denote the expected value of time-discounted future payoffs for a type-hn investor. The value is defined implicitly by

$$rv_{hn} = \lambda_d\mu_{lo}g_{lo-hn} + \lambda_f\mu_{fo}g_{fo-hn} + \lambda_f\mu_{fe}g_{fe-hn} - \rho_d\left(v_{hn} - v_{ln}\right), \qquad \text{(v-hn)}$$

where each g_{lo-hn}, g_{fo-hn}, and g_{fe-hn} denotes the investor's **gain from trade** (in fact, an equal share of the gain, which we define later). The meeting rate for direct trading, taking into account the population of sellers, is $\lambda_d\mu_{lo}$, and the gain from trade is g_{lo-hn}. Two other terms are defined similarly for the cases of trading with either a fo- or fe-type fund. The investor changes its type from high to low with rate ρ_d, in which case it loses value equivalent to $v_{hn} - v_{ln}$.

[12] If all eigenvalues of the linearized system at the steady-state solution have negative real parts, then the solution is asymptotically stable (Hirsch and Smale, 1973).

The values for other types are defined similarly as follows:

$$rv_{ho} = u_h - \rho_d (v_{ho} - v_{lo}), \tag{v-ho}$$

$$rv_{ln} = \rho_u (v_{hn} - v_{ln}), \tag{v-ln}$$

$$rv_{lo} = u_l + \lambda_d \mu_{hn} g_{lo-hn} + \lambda_f \mu_{fn} g_{lo-fn} + \rho_u (v_{ho} - v_{lo}), \tag{v-lo}$$

$$rv_{fn} = \lambda_f \mu_{lo} g_{lo-fn} + \lambda_s \mu_{fe} g_{fe-fn}, \tag{v-fn}$$

$$rv_{fo} = u_f + \lambda_f \mu_{hn} g_{fo-hn} - \rho_e (v_{fo} - v_{fe}), \tag{v-fo}$$

$$rv_{fe} = u_e + \lambda_f \mu_{hn} g_{fe-hn} + \lambda_s \mu_{fn} g_{fe-fn}. \tag{v-fe}$$

Note that payoff flows are included for owner types. (the payoff flow is zero for non-owners).

We assume that the transaction prices (that we will characterize in the next subsection) will be set to ensure an equal division of gain from trade between a buyer and a seller.[13] Then, each agent's share of the trade gain is:

$$\begin{aligned}
g_{lo-hn} &\equiv (1/2)(v_{ho} + v_{ln} - v_{lo} - v_{hn}),\\
g_{lo-fn} &\equiv (1/2)(v_{fo} + v_{ln} - v_{lo} - v_{fn}),\\
g_{fo-hn} &\equiv (1/2)(v_{ho} + v_{fn} - v_{fo} - v_{hn}),\\
g_{fe-hn} &\equiv (1/2)(v_{ho} + v_{fn} - v_{fe} - v_{hn}),\\
g_{fe-fn} &\equiv (1/2)(v_{fo} + v_{fn} - v_{fe} - v_{fn}) = (1/2)(v_{fo} - v_{fe}).
\end{aligned}$$

Let $V(\theta)$ denote the above system of value equations (v-hn)-(v-fe), with μ being replaced by the unique steady-state solution of $P(\theta)$.

Proposition 2. *There exists a unique solution of $V(\theta)$.*

We find the unique steady-state population measures μ and the values v, assuming that all tradings are active. If the unique steady-state solution (μ, v) results in positive trade gains, we call it a **steady-state equilibrium**.[14] The intuition behind the conditions for positive

[13] Our qualitative results do not depend on the assumption of equal bargaining power. Ahern (2012) observes that the dollar gains of trades are often equally split between buyers and sellers.

[14] A unique steady-state equilibrium appears in Duffie et al. (2005), but multiple equilibria appear more commonly with financial market applications: e.g., Vayanos and Weill (2008) and Trejos and Wright (2016).

trade gains ((21) and (22) in Appendix) is as follows. The trade gain g_{fe-fn} is trivially positive from $u_f > u_e$: secondary transactions bail out funds under liquidity constraints. The gains from investors' direct trading and fund-investor trading are related to each other as $g_{lo-hn} = g_{lo-fn} + g_{fo-hn}$. This implies that both direct trades and indirect trades through fund intermediation result in the same total gains. Similarly, gains in fund-investor trading and secondary trading are related as $g_{fe-hn} = g_{fo-hn} + g_{fe-fn}$. Hence, it is sufficient to ensure that g_{lo-fn} and g_{fo-hn} are positive. This can be achieved, for example, by having a significant difference between u_f and u_l (indicating a substantial contribution by buyout funds to operational efficiencies of assets), as well as a notable difference between u_h and u_f (substantial benefits from synergies for certain investors).

The transaction prices are determined so that buyers and sellers equally share the gains from trades (i.e., the equal bargaining power assumption):

$$
\begin{aligned}
p_{lo-hn} &\equiv (1/2)(v_{ho} + v_{lo} - v_{hn} - v_{ln}),\\
p_{lo-fn} &\equiv (1/2)(v_{fo} + v_{lo} - v_{ln} - v_{fn}),\\
p_{fo-hn} &\equiv (1/2)(v_{ho} + v_{fo} - v_{fn} - v_{hn}),\\
p_{fe-hn} &\equiv (1/2)(v_{ho} + v_{fe} - v_{fn} - v_{hn}),\\
p_{fe-fn} &\equiv (1/2)(v_{fo} + v_{fe} - 2v_{fn}).
\end{aligned}
$$

Finally, the **social welfare** is the sum of the values of all player types weighted by the populations in the economy, $W \equiv \sum_{i \in \mathcal{T}} \mu_i v_i$, which is a standard definition in the literature (see, for example, Duffie et al. (2005) and Hugonnier et al. (2020)).

4 Calibration

We calibrate our model to the US M&A middle market intermediated by the PE middle-market buyout funds. The middle-market (MM) corporate acquisitions lend itself naturally to the search framework. A typical transaction involves a sale of assets—either all the corporation's assets or a subdivision—by corporate investors or by PE buyout funds. The process takes about a year for sellers to find appropriate buyers and close transactions (Boone and Mulherin, 2011). A PE buyout fund acquires a small number of portfolio firms, holds them as inventory, adds operational value through better management, and exits by selling the portfolio firms.[15]

Our main data set is sourced from Pitchbook, a leading provider of private deal activity data. The data considers transactions between \$25 million and \$1 billion as MM deals, and buyout PE funds with capital commitments between \$100 million and \$5 billion as MM PE funds. These funds primarily acquire MM companies.[16] Table 1 presents MM PE fund counts, US MM M&A deal totals, and MM PE deals from 2007 to 2017. It categorizes MM PE exits by exit type: corporate acquisitions (η_{lo-hn}) or SBOs (η_{fe-fn}). IPOs represent a minor portion of PE exits and are thus excluded from our model, with IPOs counted the same as corporate acquisitions in the data. The number of PE MM funds has been increasing significantly with a compounded growth rate of 10.4% from 2000 (293 funds) to 2021 (2,472 funds).[17]

We obtain the number of mid-sized companies in the US to be 102,626 from the 2012 US Economic Census Data.[18] Accordingly, we calibrated our model to align with the US middle market data for 2012. We use data from other years to evaluate how accurately our model, which was calibrated using 2012 data, predicts PE activities in other years.

Given the normalization $k_v = 1$, the number of MM PE funds in 2012 implies $k_f = 1,571/102,626 \simeq 0.015$. Moreover, 1,925 out of 9,276 M&A deals are by PE funds, which leaves $\eta_{lo-hn} = 7,351$ corporate direct acquisitions. Lastly, 45% (409 out of 904=464+31+409) of PE exits are by SBOs.

In what follows, we make certain adjustments to the raw Pitchbook data to approximate our steady state equilibrium conditions. PE activities in the mid-size M&A market surged significantly in the late 2000s (refer to Table 1). Accordingly, when PE funds purchased

[15]We differentiate PE buyout funds from other PE funds, such as venture capital funds, which typically invest in fractional equity stakes of start-ups and early-stage firms.

[16]For more details about the methodology, see https://pitchbook.com/news/articles/pitchbook-report-methodologies

[17] PitchBook data shows that there were 293 PE MM funds in 2000, and the number of PE funds changed from 2,176, 2,269, and 2,361 to 2,472 between 2018 and 2021, respectively.

[18]Mid-size companies are those with annual revenues between \$20 million and \$1,000 million. We exclude small companies from our analysis due to the lack of reliable data on their acquisition activities and omit large companies to maintain homogeneity in our sample set.

	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017
The number of middle-market (MM) PE funds ($100M-$5B)											
	882	987	1,322	1,378	1,489	1,571	1,691	1,721	1,815	1,921	1,994
The number of MM PE exits by type											
(Corporate Acquisition)	388	287	166	348	405	464	411	546	533	479	463
(IPO)	42	8	19	33	21	31	33	42	24	19	21
(Secondary Buyout)	318	186	91	277	303	409	364	488	518	497	511
The percentage of SBOs among PE exits											
	43%	39%	33%	42%	42%	45%	45%	45%	48%	50%	51%
The number of PE MM deals ($25M-$1B)											
	1,876	1,298	744	1,337	1,515	1,925	1,730	2,267	2,265	2,405	2,509
The total number of MM M&A deals ($25M-$1B)											
	8,072	7,009	5,514	7,100	8,290	9,276	8,761	11,124	12,064	11,352	11,735

Table 1: Pitchbook data on the number of PE MM funds (capital commitment between $100M and $5B), the number of MM PE fund exits by type, the total number of M&A MM deals ($25M-$1B), and the number of deals by PE from 2007 to 2017. The numbers of PE MM funds for later years are shown in footnote 17.

assets in 2012, SBO transaction opportunities likely fell short of the figure that would arise in a (hypothetical) steady state, leading the funds to seek more PBOs instead. Hence, we choose to take only the percentage of PE exits by SBOs (45%), which was more stable than the numbers of PBOs and SBOs around 2012. Then, numbers of PBOs and SBOs implied by the steady-state equilibrium condition and the total number of PE deals (1,925) are 683 and 559, respectively.[19]

Table 2 presents the key statistics we match the calibrated model to the data. First, the average time to sell for corporate investors and PE funds is from the average of the values provided in online reports prepared by selling agents, such as business brokers or investment bankers (see Appendix B for the list of references). On average, it takes approximately 11 months (0.91 years) for PE funds and 15 months (1.25 years) for corporate investors

[19] The details are as follows. The total number of PE deals, 1,925, includes PE acquisitions (PBOs), SBOs, and PE exits other than SBOs ($1,925 = \eta_{lo-fn} + \eta_{fe-fn} + (\eta_{fe-hn} + \eta_{fo-hn})$). Additionally, SBOs contribute 45% of PE exits ($\frac{\eta_{fe-fn}}{\eta_{fe-fn} + (\eta_{fe-hn} + \eta_{fo-hn})} = 45\%$). Since the number of primary buyouts (PBOs) equals PE exits other than SBOs ($\eta_{lo-fn} = \eta_{fe-hn} + \eta_{fo-hn}$) in steady state, the inferred number of SBOs and PBOs must be $\eta_{fe-fn} = 559$ and $\eta_{lo-fn} = 683$, respectively.

to complete a firm sale.[20] Second, for the fund performance, we use the Public Market Equivalent (PME) introduced by Kaplan and Schoar (2005) and Sorensen and Jagannathan (2015). We consider the PME to be the average of 1.01 from various estimates.[21] The EV/EBITDA multiple is set equal to the average of 9.0 from 2005 to 2017 from a recent report by FactSet Research Systems Inc.[22]

Description	**Model Statistic (for normalized values)**	**Data**	**Data Source**
Corporate (Direct) acquisitions	$\eta_{lo-hn} = \lambda_d \mu_{lo} \mu_{hn}$	7351	
Primary Buyouts (PBOs)	$\eta_{lo-fn} = \lambda_f \mu_{lo} \mu_{fn}$	683	Pitchbook Inc.
Secondary Buyouts (SBOs)	$\eta_{fn-fe} = \lambda_s \mu_{fn} \mu_{fe}$	559	
Avg time to sell for investors	$E[\tau_{sv}]$ (eq. (4))	1.25 years	Various references
Avg time to sell for PE funds	$E[\tau_{sf}]$ (eq. (5))	0.91 years	(Table 7)
Fund performance (PME)	PME (eq. (6))	1.01	Various references
Price multiple (EV/EBITDA)	P_{lo-hn}/u_l	9.0	(see main text)

Table 2: Key statistics on the corporate acquisition market. Each row of the table contains a model statistic that we match with data for calibration. Note that we use the steady-state inferred numbers for PBOs and SBOs (fn. 19).

Next, we examine the remaining model parameters listed in top panel of Table 3. Some of these parameters are directly observed in the existing literature. Low-type corporate investors' flow payoff is normalized as $u_l = 1$. We set $u_h = 1.4$ from Betton et al. (2008), which report an average 43% takeover premium over 4,880 acquisitions during 1980-2002, and Bargeron et al. (2008), which find that the takeover premium paid by a private acquirer is 40.9%. We set $u_f = 1.2$ from Guo et al. (2011), which estimated a median gain of 14.3%

[20]Time to sell includes time taken in the preparation process and the listing-to-sale process. The preparation process for PE funds takes only an average of 2 months – much shorter than an average of 6 months for corporate investors. Portfolio firms of PE funds are usually in a better state of readiness to approach the market due to high-quality governance, accounting, and information systems. The listing-to-sale process takes an average of 9 months for selling agents.

[21]Public Market Equivalent (PME) is defined as the ratio of cash outflows over cash contributions, both discounted at the public market total return (e.g., S&P 500 index) after subtracting management fees paid to the fund managers. Kaplan and Schoar (2005) estimate an average PME of 0.93 for PE funds in the period 1980-1994, while Phalippou and Gottschalg (2008), using a similar dataset but different methodology, report an average PME of 0.88. Harris et al. (2014), on the other hand, reports significantly better performance with an average PME of 1.22 for the period 1984-2008. The estimates of PME by PitchBook Data, Inc. yield an average of 1.00 for the period 2006-15.

[22]`https://www.factset.com/hubfs/mergerstat_em/monthly/US-Flashwire-Monthly.pdf`.

	Parameters	Variable	Value
	No. of corporate investors	k_v	1.0
	No. of PE funds	k_f	0.015
	No. of assets	k_a	0.5
(Observed)	Flow Payoff low type	u_l	1
	Flow payoff high type	u_h	1.4
	Flow payoff PE (harvesting)	u_f	1.2
	Flow payoff PE (exiting)	u_e	1.03
	Low valuation shock	ρ_d	0.25
	High valuation shock	ρ_u	0.18
	Liquidity shock	ρ_e	0.71
(Estimated)	Match intensity (direct trading)	λ_d	19.7
	Match intensity (PBO)	λ_f	16.13
	Match intensity (SBO)	λ_s	700
	Discount rate	r	11.8%

Table 3: Fitted parameters of calibration. Some parameters are directly observed from data (in the upper table), while others (in the lower table) are selected to ensure that the steady-state equilibrium best matches the data in Table 2.

by large-market funds, with an adjustment to 20% because we focus on mid-market funds associated with higher risk and higher returns, as opposed to large-market funds.[23] For the flow payoff net of liquidity cost u_e, Nadauld et al. (2016) find that fund investors under liquidity shocks sell their PE ownership to other fund investors at a 13.8% discount. This observation motivates our choice of $u_e = (1 - 0.138) \times u_f \simeq 1.03$.[24]

We do not directly observe the number of assets, so our benchmark analysis assumes $k_a = 0.5$, but the calibration results vary insignificantly if $k_a = 0.25$ and $k_a = 0.75$.

We estimate the parameters (ρ, λ, r) in the bottom panel of Table 3 that offer the best

[23]In earlier version, we chose $u_f = 1.3$ from Kaplan (1989) which reported 45.5%, 72.5%, and 28.3% increases in net cash flow/sales each year for the first three years following the buyout, and Muscarella and Vetsuypens (1990), Opler (1992), and Andrade and Kaplan (1998), which estimated the increase in operating profits of target firms after fund buyouts as 23.5%, 16.5%, and 52.9%, respectively. However, the private equity industry may have undergone transformations over the years (Strömberg, 2008), and the profitability effects of large public-to-private deals have weakened (Guo et al., 2011).

[24]Indeed, the equilibrium trade patterns, such as trade volumes, population distributions, and time to sell or buy, remain essentially invariant with respect to the flow-payoff parameters. We will discuss the sensitivity of other statistics such as value, welfare, and prices in later sections.

fit to the statistics in Table 2. Specifically, each choice of the remaining parameters' values, together with the directly observed parameters (k, u), defines a market $\theta = (k, r, u, \rho, \lambda)$. We compute the statistics $Y_i(\rho, \lambda, r; k, u)$ for each row $i = 1, \ldots, 7$ in Table 2, using the closed-form expressions for various metrics, such as the average time to sell (obtained in Proposition 9), PME (obtained in Proposition 11), EV/EBITDA (P_{lo-hn}/u_l), and compare them with the observed data Y_i^{obs}.

We obtain estimates for (ρ, λ, r) by minimizing the sum of squared residuals (SSR) subject to positive trade gains in the unique steady-state solution of the market $(\rho, \lambda, r; k, u)$, i.e., $\min_{\rho,\lambda,r} \sum_{i=1}^{7} \left(\frac{Y_i(\rho,\lambda,r;k,u) - Y_i^{obs}}{Y_i^{obs}} \right)^2$ subject to $g_m(\beta; k, u) \geq 0$, for each $m \in \mathcal{M}$. Our model fits the observed data with a high degree of accuracy. The minimum SSR is approximately 5×10^{-4}.

The lower section of Table 3 summarizes the parameter estimates.[25] The parameter estimates are of reasonable magnitudes. The estimated type transition rates ρ_u and ρ_d suggest that the corporate investors' type transitions take on average about $1/\rho_u = 5.40$ years from low to high, and $1/\rho_d = 4.05$ years from high to low. Moreover, a PE fund holding an asset can expect to experience a liquidity shock approximately every $1/\rho_e = 1.41$ years. The matching intensities $(\lambda_d, \lambda_f, \lambda_s)$ are not directly interpretable, but, for example, $\frac{1}{\lambda_d \mu_{lo}} = 0.5$ is the average time, in years, for a high-type investor to meet a low type and make a direct transaction. Lastly, the estimated discount rate $r = 12.5\%$, although high, seems reasonable given that assets represent stakes in mid-size private firms.

In the following sections, we address various implications of our calibration, including: (i) Is there an oversupply or undersupply of tradable assets relative to the number of corporate buyers and PE funds? (ii) How are fund valuations affected by liquidity provision through SBOs and operational improvements made by fund managers? (iii) How does PE entry impact fund valuation and transaction prices? (iv) What are the welfare losses associated with search frictions?

[25] The calibrated search rates tend to be very large due to our normalization of $k_v = 1$ and motivates us to study a fast-search market. See the fast search market analysis in Section 6 for detailed discussions.

5 Equilibrium Analysis of the Secondary Market

We analyze equilibrium with a focus on the secondary market, incorporating both qualitative and quantitative aspects through model calibration.

We find that secondary trading enhances liquidity and improves welfare, as expected. Each secondary trade bails a fund out of liquidity constraints and offers fund buyers more transaction opportunities, at no cost to any other types of agents. Thus, a more liquid secondary market attenuates the effects of fund liquidity shocks and improves overall welfare.

Proposition 3. *1. While liquidity shocks reduce funds' values ($v_{fe} < v_{fo}$), their influence is mitigated by more liquid secondary market ($\frac{\partial(v_{fo}-v_{fe})}{\partial\lambda_s} < 0$) and vanishes when the secondary market becomes completely liquid ($\lim_{\lambda_s\to\infty}(v_{fo} - v_{fe}) = 0$).*

2. The welfare increases in the secondary market liquidity ($\frac{\partial W}{\partial\lambda_s} = \frac{\partial\mu_{fo}}{\partial\lambda_s}(\frac{u_f-u_e}{r}) > 0$).

A faster secondary market enables funds under selling pressure (fe) to exit and restart their life cycle as fn swiftly, while more funds in the investment phase (fn) transition to the harvesting phase (fo). Each unit measure of secondary transactions shifts population from μ_{fe} to μ_{fo} without altering μ_{fn}, resulting in increased welfare by the value of increased payoff flow with time discount $\frac{u_f-u_e}{r}$.

Quantitatively, in the corporate acquisition market, the impact of secondary buyouts on overall welfare (Proposition 3) is limited, given the small number of funds compared to corporate investors ($k_f = 0.015$). For instance, 1% increase in secondary market liquidity from $\lambda_s = 700$ would increase the number of secondary buyouts by at most 3.6 per year, resulting in a mere 0.03% increase in the total welfare ($\frac{\partial W}{\partial\lambda_s}\frac{\lambda_s}{W} = 0.03$).[26]

Self-interested funds support each other through secondary trades, allowing funds under selling pressure to exit more quickly. Additionally, fund buyers benefit from increased trade opportunities when funds look to exit. A higher search rate in the secondary market

[26]In equilibrium, the number of secondary buyouts is given by $\eta_{fn-fe} = \lambda_s\mu_{fn}\mu_{fe}$. To estimate the impact of a 1% increase in the search rate, $\lambda_s = 700$, we multiply it by $\mu_{fn} = 0.0024$, $\mu_{fe} = 0.0021$, and the number of corporate investors (102,626) to account for normalization ($k_v = 1$). This calculation provides an upper bound on the increase in the number of secondary buyouts since an increase in the search rate λ_s reduces the number of funds under selling pressure μ_{fe} and the number of fund buyers μ_{fn}.

strengthens this channel, enabling additional funds to enhance their expected value at the time of the new life cycle (v_{fn}).

Proposition 4. *There exists $\overline{k}_f$ such that if $k_f < \overline{k}_f$, then there is a complementarity between the secondary market liquidity and the number of funds ($\frac{\partial^2 v_{fn}}{\partial \lambda_s \partial k_f} > 0$).*

Furthermore, the mutual benefits resulting from secondary trades among funds can be significant enough to outweigh the value reduction caused by narrower buy-sell spreads due to increased competition. Specifically, when the secondary market search rate (λ_s) is high, the value of each fund at the time of a new life cycle increases in the number of funds present.

Proposition 5. *There exists $\overline{k}_f$ and a function $\overline{\lambda}_s(k_f)$ such that, if there are not too many funds ($k_f < \overline{k}_f$) and the secondary market is liquid enough ($\lambda_s > \overline{\lambda}_s(k_f)$), then funds' value increases in their number ($\frac{\partial v_{fn}}{\partial k_f} > 0$).*[27]

The proof of Propositions 4 and 5 obtains a closed-form expression for the marginal value ($\frac{\partial v_{fn}}{\partial k_f}$) when $k_f \approx 0$ and demonstrates its increase in the secondary-market search rate λ_s.

To gain insights into the quantitative implications of Propositions 4 and 5, we analyze the expected value of a new fund (v_{fn}) in our model's calibration. In Figure 2, we present the value of a new fund (v_{fn}) across various numbers of funds (k_f) and different levels of secondary market search rate (λ_s). The other parameters are kept constant at their calibrated values. The vertical line at k_f^* and the curve for λ_s^* correspond to the observed number of funds and the calibrated search rate for secondary trades, respectively.

Intuitively, when the market consists of a small number of funds, the addition of a new fund has a positive impact on the expected value of each fund, confirming Propositions 4 and 5. However, as the number of funds continues to rise, competition among them becomes more intense, particularly in the pursuit of intermediation opportunities. Consequently, this heightened competition gradually offsets the benefits arising from complementarities.

[27] The matching technology plays only a limited role in Proposition 5. Note that in our model, the rate of meetings between funds of type fe and fn is $\lambda_s \mu_{fn} \mu_{fe}$, and so scaling up the total number of funds k_f by x while holding the type distribution fixed scales up the meeting rate by x^2 and the individual fund's meeting intensities by x. The qualitative result ($\frac{\partial v_{fn}}{\partial k_f} > 0$) holds in general. Observe that the derivative $\frac{\partial v_{fn}}{\partial k_f}$ at $k_f \approx 0$ in Figure 2 gets indefinitely large as λ_s increases. For any matching technology with increasing returns to scale, if λ_s is sufficiently large and k_f is small, then $\frac{\partial v_{fn}}{\partial k_f} > 0$.

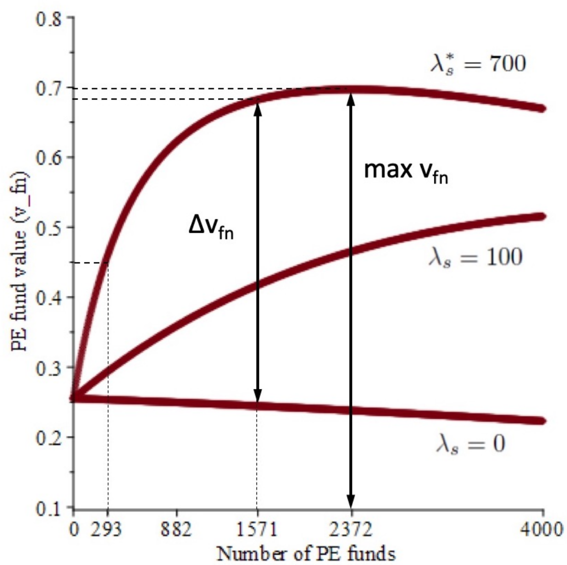

Figure 2: Each graph displays a new fund's expected value v_{fn} for varying numbers of funds. The top curve represents the expected value with the calibrated secondary-market search rate $\lambda_s = 700$, while the bottom two curves reflect counterfactual values of λ_s. All other parameters remain fixed at calibrated values. In 2012, there were 1,571 PE funds ($k_f = \frac{1{,}571}{102{,}626} \simeq 0.015$). The secondary market, specifically an increase in λ_s from 0 to 700, corresponds to 64% of fund valuations ($\frac{\Delta v_{fn}}{v_{fn}} \simeq 0.64$). The peak valuation v_{fn} occurs when the number of funds reaches 2,372. The fund value v_{fn} increases by 46% with the increase in the number of funds from 293 in 2000 to 1,571 in 2012.

With the calibrated parameters using the corporate acquisition market, the PE fund value (v_{fn}) would increase with the number of funds (k_f). This relationship is evident in Figure 2, where v_{fn} demonstrates an increasing trend for the calibrated search rate λ_s^* with respect to k_f. The number of PE MM funds has dramatically increased from 293 to 1,571 between 2000 and 2012. Our calibration, based on 2012 data, shows that this increase in the number of funds, holding everything else constant, is responsible for a 46% increase in fund valuation. The analysis also indicates that this trend of increase would be exhausted when the number of funds reaches 2,371, corresponding to the peak value of $v_{fn} = 0.697$. A further increase in the number of funds is likely to create pressure on fund valuation due to increased competition among funds. Interestingly, the number of PE funds decreased from 2,472 in 2021 to 1,838 in 2023.

Next, we perform a counterfactual exercise of shutting down the secondary market. The exercise shows that PE funds could lose 64% of their expected value if they were not allowed to engage in secondary trades. Despite acknowledging criticisms against SBOs, our results show that SBOs contribute, rather than detract, to PE funds generating high returns.

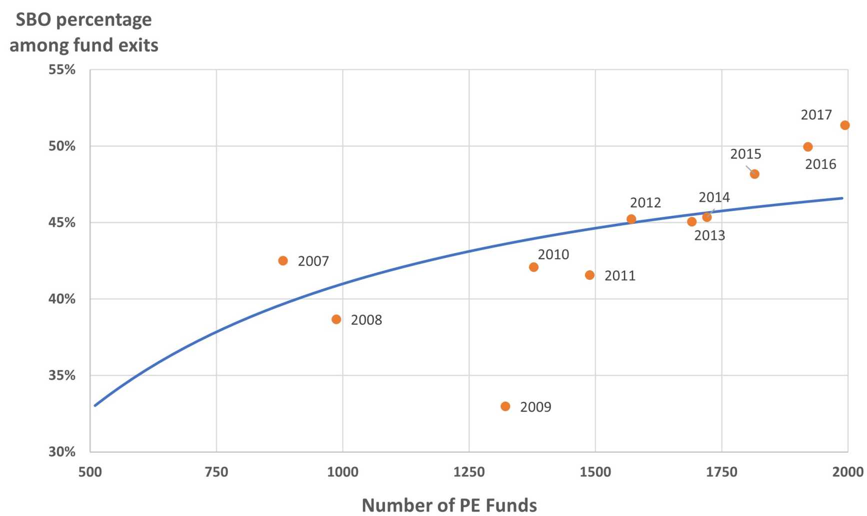

Figure 3: The solid line presents the share of secondary buyouts exits (%SBO) (y-axis) predicted by our model for different numbers of funds (x-axis). The %SBO values are obtained from the model predictions based on the calibrated parameters from Table 3, just changing the number of funds k_f. The dots represent the actual number of PE funds and the corresponding share of SBO transactions from 2007 to 2017 from the Pitchbook data in Table 1.

A further quantitative result is on funds exits through SBOs. When the number of funds increases, an increasing fraction of funds exits through secondary trading rather than sales of assets to investors (Figure 3). While it is evident that the number of secondary trades increases with the growth in funds, our model explains a tandem increase in the *share* of exits through secondary trades. The solid line in Figure 3 represents the predicted share of secondary buyouts exits (%SBO in the y-axis) by our model, based on our calibration using 2012 data, with different numbers of PE funds (in the x-axis). The figure shows that with an increase (decrease) in the number of PE funds, there is a corresponding increase (decrease) in the %SBO. The predicted %SBO by the calibrated model (the solid line) approximates well the actual numbers (the scatter plots) of PE funds and the percentages of SBO exits from 2007 to 2017 in Table 1.

The quantitative results hinge on selected parameter values in the upper part of Table 3. The model calibration remains largely unaffected by our choices of u_f and discount D ($u_e = (1-D)u_f$). The estimated parameters, as seen at the bottom of Table 3, primarily represent trade intensities ($\lambda_d, \lambda_f, \lambda_s$) and type changes ($\rho_u, \rho_d, \rho_e$). These parameters determine steady-state populations (μ-hn)-(μ-fe) and trade volumes, which are captured by Pitchbook data. However, our choices of u_f and u_e primarily influence the fund valuation v_{fn}. Revision to our quantitative results regarding v_{fn} may be necessary if readers find other parameter values more suitable. A 1% improvement in firms' operation u_f leads to a significant 8.2% increase in a new fund's value from $v_{fn} = 0.65$. However, a similar improvement in u_e has a negligible influence on fund value, with a sensitivity of only 1.02 due to the presence of a vibrant SBO market.[28, 29]

[28]The sensitivities of the values to cash flow parameters are as follows:

$$\left[\frac{\partial x_i}{\partial \theta_j}\frac{\theta_j}{x_i}\right]_{i,j} = \left[\begin{array}{c|ccc} x_i \backslash \theta_j & u_h & u_f & u_e \\ \hline v_{fn} & -1.391 & 8.179 & 1.026 \\ v_{ho} & 0.706 & 0.007 & 0.007 \\ v_{ln} & 3.537 & -0.116 & -0.135 \\ v_{lo} & 0.539 & 0.011 & 0.011 \\ v_{hn} & 3.583 & -0.011 & -0.135 \end{array}\right].$$

[29]Furthermore, consider the SBO contribution to the fund valuations $\Delta v_{fn} = v_{fn}|_{\lambda_s=700} - v_{fn}|_{\lambda_s=0}$. If the liquidity discount is less than 13.8%, meaning u_e is higher than 1.03, then PE funds do not suffer from liquidity discount as much as before. Consequently, the SBO's contribution to fund valuation will be less than 64%. If we recalculate the sensitivity at $v_{fn}|_{\lambda_s=0} = 0.23$ (assuming $\lambda_s = 0$), then $\frac{\partial v_{fn}}{\partial u_f}\frac{u_f}{v_{fn}} = 9.849$ and $\frac{\partial v_{fn}}{\partial u_e}\frac{u_e}{v_{fn}} = 14.036$. Since 1.026% increase from 0.65 is smaller than 14.036% increase from 0.23, choosing a higher u_e (because a lower discount D is chosen) will predict a lower SBO contribution to the fund valuation $\Delta v_{fn} = v_{fn}|_{\lambda_s=700} - v_{fn}|_{\lambda_s=0}$.

6 Other Equilibrium Properties

Corporations, PE funds and M&A practitioners keenly focus on the current and future levels of activity in M&A markets. In this section, we develop properties for the equilibrium prices, spreads, and trading speed and volume, and show how these key metrics respond to exogenous parameters, such as search frictions and transition rates. We also analyze the welfare properties of the M&A market with intermediation and demonstrate that the market exhibits search externalities. Slowing down M&A activity among corporations can improve welfare by stimulating more PE activity. This allows exit-phase funds to off-load assets, reset the life cycle, and quickly purchase new assets.

6.1 Fast Search Markets

The equilibrium analysis is sometimes obtained more easily in *fast-search* markets – i.e., economies with large search rates $\lambda = (\lambda_d, \lambda_f, \lambda_s)$. We define and derive some key properties of fast search markets in this subsection.

We set up a formal fast-search market as follows. Given any exogenous parameters $\theta \equiv (k, r, u, \rho, \lambda)$, we increase meeting rates $(\lambda_d, \lambda_f, \lambda_s)$, while preserving the relative ratios. Specifically, we consider a sequence of markets $\theta^L \equiv (k, r, u, \rho, L\lambda)$, where $L\lambda = (L\lambda_d, L\lambda_f, L\lambda_s)$, indexed by a constant L. We increase L to infinity and analyze the steady-state solution (μ^L, v^L) in the limit. To ease expositions, we assume a *regular* environment: $k_a \notin \{k_h, k_h + k_f\}$.

We show below that the fast-search market limits are well-defined as the steady-state population measures μ^L converge (the convergence of v^L, as a linear function of μ^L, follows immediately (see (16)). The speed of convergence is $O(1/L)$ which gives a precise sense of how closely a fast-search equilibrium would approximate an equilibrium of the calibrated market (in our calibration, $L \approx 103,000$):

Proposition 6. *(Convergence and Convergence Speed) Given a regular environment θ, for any $i \in \mathcal{T}$, the population limit $\mu_i^* \equiv \lim_{L\to\infty} \mu_i^L$ and the convergence speed $\mu_i^{**} \equiv \lim_{L\to\infty} L(\mu_i^L - \mu_i^*)$ exist.*

The results in fast-search markets approximate a market with many participants. The reason is that we normalized the total number of investors as $k_v = 1$ and proportionally re-scaled the number of funds and trade volumes. Let K_v be the total number of investors before normalization, with K_i for $i \in \mathcal{T}_v$ being the number of type-i investors. If each pair of investors meet at a Poisson rate l_d, the total number of direct trading (say, per year), with normalization, would be $(l_d K_{lo} K_{hn})/K_v = (l_d K_v)(K_{lo}/K_v)(K_{hn}/K_v) = (l_d K_v)\mu_{lo}\mu_{hn}$.

Markets for corporate acquisitions commonly have many buyers, sellers, and intermediaries, such as in our calibration. Therefore, the closed-form expressions we can obtain for fast search markets help us understand the quantitative equilibrium properties. In the remainder of this section, while some results are general, others are obtained only for fast search markets.

6.2 Welfare

The total welfare W can be decomposed into the sum of the welfare to the investors and funds in the economy: $W = W_v + W_f$, where $W_v \equiv \sum_{i \in \mathcal{T}_v} \mu_i v_i$ and $W_f \equiv \sum_{i \in \mathcal{T}_f} \mu_i v_i$. The welfare measures, which are defined based on the agents values, are naturally related to the investors' and funds' payoff flows and gains from trades as follows:

Proposition 7. *(Equilibrium Welfare)*

$$rW = \mu_{ho}u_h + \mu_{fo}u_f + \mu_{fe}u_e + \mu_{lo}u_l, \tag{2}$$

$$rW_v = \mu_{ho}u_h + \mu_{lo}u_l + \underbrace{\lambda_f \mu_{lo}\mu_{fn} p_{lo-fn}}_{\textit{sales to funds}} - \underbrace{\lambda_f \mu_{hn}(\mu_{fo}p_{fo-hn} + \mu_{fe}p_{fe-hn})}_{\textit{purchases from funds}}, \quad \textit{and} \tag{3}$$

$$rW_f = \mu_{fo}u_f + \mu_{fe}u_e + \underbrace{\lambda_f \mu_{fo}\mu_{hn} p_{fo-hn} + \lambda_f \mu_{fe}\mu_{hn} p_{fe-hn}}_{\textit{sales to investors}} - \underbrace{\lambda_f \mu_{fn}\mu_{lo} p_{lo-fn}}_{\textit{purchases from investors}}.$$

The first two terms for the investors' welfare W_v represent payoff flows to ho- and lo-type investors. The next term represents the inflow from selling assets to funds. Only lo-type investors sell assets to funds with the total rate $\lambda_f \mu_{lo}\mu_{fn}$ and at price p_{lo-fn}. The last term represents the hn-type investors' payments to funds: p_{fo-hn} to fo-type funds with the aggregate rate of $\lambda_f \mu_{hn}\mu_{fo}$, or p_{fe-hn} to fe-type funds with the aggregate rate of $\lambda_f \mu_{hn}\mu_{fe}$. A similar interpretation explains the funds' welfare W_f.

Inefficiency and Search Externality To study the inefficiencies created by search externalities, we turn to the fast-search equilibrium welfare $W^* \equiv \lim_{L\to\infty} W^L$. According to Proposition 7, $W^* = \mu^*_{ho}u_h + \mu^*_{fo}u_f + \mu^*_{fe}u_e + \mu^*_{lo}u_l$, where $\mu^*_i \equiv \lim_{L\to\infty} \mu^L_i$, and the welfare W^L converges to W^* at the same speed $O(1/L)$ as μ^L (Proposition 6).

We compare the fast-search equilibrium welfare W^* against two extreme situations: an autarkic economy with no functioning market or fund intermediation, and a centralized economy with a planner moving assets across agents without search friction. In the autarkic economy, a k_a fraction among $k_v(= 1)$ corporations hold assets with no trades, resulting in the welfare $\underline{W}$ such that $r\underline{W} = k_a(k_h u_h + k_l u_l)$. In the centralized economy, a planner solves

$$r\overline{W} \equiv \max_{\mu \in \mathbb{R}^{\mathcal{T}}_+} \mu_{ho}u_h + \mu_{fo}u_f + \mu_{fe}u_e + \mu_{lo}u_l,$$
$$\text{subject to} \quad \mu_{ho} + \mu_{hn} = k_h, \quad \mu_{lo} + \mu_{ln} = k_l, \quad \text{and} \quad (1).$$

The maximum welfare $\overline{W}$ takes into account exogenous type changes ρ_u and ρ_d but ignores search frictions. The liquidity shock ρ_e imposes no restriction on the planner who can transfer assets between funds instantaneously. The planner can set aside an ϵ mass of funds as type fn and transfer assets to them when some other funds receive a liquidity shock. The mass of fo and fe type funds remain the same. Consequently, the mass arbitrarily close to k_f of assets can be held by fo type funds.

The population measures $\overline{\mu}$ that achieves the maximum welfare is such that $\overline{\mu}_{ho} = \min\{k_a, k_h\}$, $\overline{\mu}_{fo} = \min\{(k_a - k_h)^+, k_f\}$, $\overline{\mu}_{lo} = (k_a - k_f - k_h)^+$, and $\overline{\mu}_i = 0$ for $i \neq ho, fo, lo$. In essence, assets are allocated to high-type investors up to their steady-state population k_h; any remaining assets are given to funds up to k_f; and the still-remaining assets are given to low-type investors. The maximum welfare satisfies $r\overline{W} = \overline{\mu}_{ho}u_h + \overline{\mu}_{fo}u_f + \overline{\mu}_{lo}u_l$.[30]

We compare the efficient allocation $\overline{\mu}$ (the upper part of Table 4) with the fast-search equilibrium population μ^* (the lower part of Table 4).

[30] The maximum welfare $\overline{W}$ is also achieved by a planner who is under the search friction, like agents, but can choose not to execute some transactions. The planner's problem is $rW_p(\lambda) \equiv \sup_{0\leq\lambda_p\leq\lambda} \mu_{ho}u_h + \mu_{fo}u_f + \mu_{fe}u_e + \mu_{lo}u_l$, subject to μ being a solution of $P(k, r, u, \rho, \lambda_p)$. Fast search allows the planner to achieve the maximum welfare approximately: i.e., $\overline{W} = \lim_{\lambda\to\infty} W_p(\lambda)$. Intuition will become clear after Proposition 8. The planner slows down the investors' direct trading, eliminates search externalities, and increases welfare to the maximum.

	A. $k_a < k_h$	B. $k_h < k_a < k_h + k_f$	C. $k_h + k_f < k_a$
$\overline{\mu}_{ho} =$	k_a	k_h	k_h
$\overline{\mu}_{fo} =$	0	$k_a - k_h$	k_f
$\overline{\mu}_{lo} =$	0	0	$k_a - k_f - k_h$
$\overline{\mu}_{fe} =$	0	0	0
$\mu^*_{ho} =$	k_a	k_h	k_h
$\mu^*_{fo} =$	0	$k_a - k_h$	$< k_f$
$\mu^*_{lo} =$	0	0	$k_a - k_f - k_h$
$\mu^*_{fe} =$	0	0	> 0

Table 4: The population under fast search (μ^*) and the efficient allocation ($\bar{\mu}$).

Proposition 8. *(Fast-search Market: Welfare) As $L \to \infty$, $W^* \equiv \lim_{L\to\infty} W^L$:*

A. *If $k_a < k_h$, then $W^* = \overline{W}$, which is independent of u_f, u_e, λ_s, and λ_d.*

B. *If $k_h < k_a < k_h + k_f$, then $W^* = \overline{W}$, which is strictly increasing in u_f and independent of u_e, λ_s, and λ_d.*

C. *If $k_h + k_f < k_a$, then W^* is strictly less than $\overline{W}$, strictly increasing in u_f, u_e, and λ_s, and strictly decreasing in λ_d.*

The characterization of the fast-search equilibrium welfare depends on the number of assets (k_a) relative to the number of potential buyers (k_h, k_f). A sufficiently large number of assets ($k_a > k_h + k_f$) gives rise to an inefficient fast-search equilibrium. The calibrated ρ_u and ρ_d imply $k_h \equiv \frac{\rho_u}{\rho_u+\rho_d} = 0.40$ and an excess supply of assets $k_a > k_h + k_f$. The result, together with large meeting rates, suggests that the US corporate acquisition market is close to Case C of the fast-search market.[31]

Suppose that the fast-search market has sufficiently many potential buyers ($k_a < k_h + k_f$) as in Cases A and B (the first two columns in Table 4). Fast search allows investors and funds to quickly transfer assets from low-type investors (lo) and exiting funds (fe) to high-type investors (hn) and, in Case B, also to funds at the investment phase (fn). Accordingly,

[31] The calibration result on the oversupply of assets is not sensitive to our choice of $k_a = 0.5$. While a choice of $k_a = 0.25$ or 0.75 results in different estimates of ρ_u and ρ_d, the oversupply remains about the same ($k_a - (k_h + k_f) \approx 0.08$).

the steady-state population μ^* equals the efficient allocation $\overline{\mu}$ and achieves the maximum welfare ($W^* = \overline{W}$).

The comparative statics of the welfare becomes trivial: the maximum welfare is dependent on payoff flows (e.g., u_f) only if the corresponding type's population (resp., μ_{fo}) is non-zero. In either case, the impact of a liquidity shock is zero. Funds transfer assets without holding any inventory (Case A) just like, e.g., in Rubinstein and Wolinsky (1987); or, they hold assets, but funds under liquidity constraints transfer assets to others through speedy secondary trades (Case B), as is the case in Duffie et al. (2005). Under a surplus of tradable assets relative to potential buyers ($k_h + k_f < k_a$), as in Case C (the third column in Table 4), the equilibrium is more interesting because, counter-intuitively, slowing down investors' direct trading improves the welfare ($\frac{\partial W^*}{\partial \lambda_d} < 0$). Since investors or funds on demand can quickly find sellers and purchase assets, there are negligible left-over high-type investors or fund non-owners. Hence, a significant fraction of exiting funds (fe) will find it difficult to offload their assets, and the welfare loss is $r(\overline{W} - W^*) = \mu^*_{fe}(u_f - u_e) > 0$.

In our calibration, the welfare gain by asset reallocations is 12.4%, relative to the autarkic situation welfare $\underline{W}$ (see page 26). This welfare gain ($W - \underline{W}$) attains 81.7% of the best possible gain ($\overline{W} - \underline{W}$). This fraction is lower than the gain in OTC markets for municipal bonds as described in Hugonnier et al. (2020), likely due to higher search frictions in the corporate acquisition market. The corporate investors' percentage share of this welfare gain is 81.2%, which leaves 18.8% to PE funds. The PE funds' welfare share is very large relative to their small number $k_f = 0.015$.

The inefficiency is a result of investors' search externalities on funds. A direct-trading by investors takes away selling opportunities from exit-phase funds and leads them to suffer from liquidity constraints for a long period of time. If the investors' direct trading were absent, an exit-phase fund could offload an asset, reset its type, and purchase another asset, all quickly under fast search. This alternative scenario results in a more efficient asset allocation.

6.3 Trading Speed, Volumes, and Prices

Average Time to Sell Statistics about the average time on the market before deal closing are widely available. We obtain below the closed-form expressions for the average time to sell for investors or funds. Those expressions are used to calibrate the model parameters.

Proposition 9. *(Time to Sell) Let τ_{sv} and τ_{sf} denote the time to sell for investors and funds. Then,*

$$E[\tau_{sv}] = \frac{1}{\lambda_d \mu_{hn} + \lambda_f \mu_{fn}}, \tag{4}$$

$$E[\tau_{sf}] = \frac{1}{\lambda_f \mu_{hn} + \rho_e} + \frac{\rho_e}{\lambda_f \mu_{hn} + \rho_e} \left(\frac{1}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn}} \right). \tag{5}$$

For example, each seller-buyer meeting arrives according to a Poisson process, so the time until the first meeting by a selling investor follows an exponential distribution with parameter $\lambda_d \mu_{hn} + \lambda_f \mu_{fn}$. A similar, but more involved, calculation gives the expected time to sell for funds.

We observe from the data that a fund typically takes around 0.91 years of search to sell an asset. However, when facing selling pressure, this time is significantly reduced to approximately 0.4 years ($\frac{1}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn}} = \frac{1}{(16.13716 \times 0.03741) + (700.3 \times 0.00516)} \approx 0.24$).

Transaction Volumes The level of activity in M&A markets is an important metric followed by in M&A practitioners. We now characterize the transaction volumes and how they respond to exogenous parameters, such as search frictions and transition rates. We obtain those results focusing on fast search markets.

For each submarket $m \in \mathcal{M}$, with a seller's type s and the buyer's type b, the **trade volume** is $\eta_m^L \equiv (L\lambda_m)\mu_s^L \mu_b^L$.

Proposition 10. *(Fast-search Market: Trade Volumes) For each submarket $m \in \mathcal{M}$, trade volume in the limit $\eta_m^* \equiv \lim_{L \to \infty} \eta_m^L$ is given by:*

	A. $k_a < k_h$	*B.* $k_h < k_a < k_h + k_f$	*C.* $k_h + k_f < k_a$
$\eta^*_{lo-hn} =$	$\lambda_d \mu^{**}_{lo} \mu^*_{hn}$	0	$\lambda_d \mu^*_{lo} \mu^{**}_{hn}$
$\eta^*_{lo-fn} =$	$\lambda_f \mu^{**}_{lo} \mu^*_{fn}$	$\lambda_f \mu^{**}_{lo} \mu^*_{fn}$	$\lambda_f \mu^*_{lo} \mu^{**}_{fn}$
$\eta^*_{fo-hn} =$	$\lambda_f \mu^{**}_{fo} \mu^*_{hn}$	$\lambda_f \mu^*_{fo} \mu^{**}_{hn}$	$\lambda_f \mu^*_{fo} \mu^{**}_{hn}$
$\eta^*_{fe-hn} =$	0	0	$\lambda_f \mu^*_{fe} \mu^{**}_{hn}$
$\eta^*_{fe-fn} =$	0	$\lambda_s \mu^{**}_{fe} \mu^*_{fn}$	$\lambda_s \mu^*_{fe} \mu^{**}_{fn}$

where (i) μ^* denotes the population limit, and (ii) for type i with $\mu_i^* = 0$, $\mu_i^{**} \equiv \lim_{L\to\infty} L\mu_i^L$ denotes the convergence speed.

With a large number of high-type investors (Case A), secondary trades are unnecessary for funds; if the deficit of high-type investors is supplemented by funds (Case B), selling investors resort to fund buyers and there are no investors' direct trading; with an excess supply of tradable assets (Case C), there are transactions in all submarkets.

The trade volumes under fast search follow from the convergence and the convergence speed of population measures (Proposition 6). For each submarket $m \in \mathcal{M}$, because of fast search, the steady-state measure of either buyers or sellers vanishes: i.e., $\mu^*_{buyer} = 0$ or $\mu^*_{seller} = 0$. If $\mu^*_{seller} = 0$, then $\eta^L_m = (L\lambda_m)\mu^L_{buyer}\mu^L_{seller} = \lambda_m \mu^L_{buyer}(L\mu^L_{seller}) \to \lambda_m \mu^*_{buyer}\mu^{**}_{seller}$ as $L \to \infty$. Proposition 10 suggests that all submarkets are active under fast search only if the market has an excess supply of tradable assets ($k_a > k_h + k_f$).

The trade volumes also identify the main drivers of the convergences of certain population measures. For example, $\mu^*_{fo} = 0$ in Case A could be due to the fact that (i) funds can rarely purchase assets because of a vanishingly small number of selling investors (lo), or (ii) funds do acquire assets, but quickly re-sell to buying investors (hn). Proposition 10 implies the latter case; funds buy/sell a significant number of assets from/to investors in the fast-search market and there are no secondary transactions (like middlemen in Rubinstein and Wolinsky (1987)). Similarly, the vanishing number of selling investors (lo) and buying investors (hn) in Case B is the result of an efficient fund-investor trading rather than an efficient market for investors' direct trading – the number of investors' direct transactions (η^*_{lo-hn}) is indeed vanishingly small.

Table 5 summarizes a comparative static analysis for trade volumes relative to search frictions and transition rates.[32] Most results are intuitive. If investors' types are more volatile (i.e., larger ρ_u and ρ_d with a fixed ratio), assets will be transferred across agents frequently, ultimately from low-type investors to high-type investors, with possible fund intermedia-

[32] The results follow directly from the expressions of μ^* and μ^{**} in Table 4 and Lemmas SA.3 and SA.4 in Supplmental Appendix, so we omit. As an example, take η^*_{lo-hn} and λ_d. Note that $\eta^*_{lo-hn} = \lambda_d \mu^*_{lo}\mu^{**}_{hn}$. In Cases A and B, $\mu^*_{lo} = 0$, so η^*_{lo-hn} is independent of λ_s. In Case C, $\mu^{**}_{hn} = \frac{\rho_u \mu^*_{ln}}{\lambda_d \mu^*_{lo} + \lambda_f(\mu^*_{fo} + \mu^*_{fe})} = \frac{\rho_u \mu^*_{ln}}{\lambda_d \mu^*_{lo} + \lambda_f k_f}$. Note that μ^*_{lo} and μ^*_{ln} are independent of λ_d. Thus, the volume η^*_{lo-hn} is increasing in λ_d because $\frac{\partial}{\partial \lambda_d}\left(\frac{\lambda_d}{\lambda_d \mu^*_{lo} + \lambda_f k_f}\right) = \lambda_f k_f > 0$.

	λ_d	λ_f	λ_s	(ρ_u, ρ_d)(with a fixed ratio)	ρ_e
η^*_{lo-hn}	$+$	$-$	0	$+$	0
η^*_{lo-fn}	$-$	$+$	0	$+$	0
η^*_{fo-hn}	$-$	$+$	$+$	$+$	$-$
η^*_{fe-hn}	$-$	$+$	$-$	$+$	$+$
η^*_{fe-fn}	$-$	$-$	$+$	$+$	$+$

Table 5: Comparative statics of trade volumes. Each $+$ (or $-$) indicates the corresponding volume to be non-decreasing (resp., non-increasing) in the parameter, and 0 indicates that the volume is independent of the parameter.

tions. Fast search among investors (i.e., higher λ_d) allows them to transact directly (i.e., higher η^*_{lo-hn}), resulting in fewer intermediation opportunities for funds. The parameters for the secondary market (λ_s and ρ_e) only shift the population measures between fo and fe. Therefore, these parameters do not affect the volume of investors' direct trading (η^*_{lo-hn}) and only shift volumes between two kinds of fund-investor transactions: η^*_{fo-hn} and η^*_{fe-hn}.

The positive response of η^*_{fe-hn} to λ_f is perhaps surprising. On one hand, a fast search between funds and investors (i.e., higher $L\lambda_f$) orchestrates more transactions between exiting funds (fe) and buying investors (hn). On the other hand, as funds are able to sell assets before receiving liquidity shocks, fewer funds enter the exit phase, which could potentially reduce the trade volume between exiting funds and buying investors. It turns out that the former effect of λ_f dominates the latter.

Spreads and Prices The performance of PE funds depends on the spread between the prices at which funds buy and sell assets. A commonly used performance measure for PE funds is the Public Market Equivalent (PME) introduced by Kaplan and Schoar (2005) and Sorensen and Jagannathan (2015)). We derive the closed-form expression of PME and use it to calibrate the model parameters.

Sorensen and Jagannathan (2015)'s PME definition is for a model discrete time with a stochastic discount. Our model is in continuous time with a deterministic discount, therefore we define PME as

$$PME \equiv \frac{\text{Present value of distributions to fund investors}}{\text{Present value of capital calls made by fund investors}} = \frac{PV_{\text{dist}}}{PV_{\text{calls}}},$$

where

$$PV_{\text{dist}} \equiv E\left[e^{-r\tau_b}\int_0^{\tau_s} e^{-rt}u(t)dt + e^{-r\tau_s}P_s\right],$$

$$PV_{\text{calls}} \equiv PV_{\text{purchasing price}} + PV_{\text{management fees}} = E\left[P_b e^{-r\tau_b}\right] + E\left[(fP_b)\int_0^{\tau_b+\tau_s} e^{-rt}dt\right].$$

A fund that does not hold an asset takes τ_b period of time until purchasing an asset at a price of P_b and takes τ_s period of time (after purchasing) until selling the asset at a price P_s.

The management fees are paid retrospectively as if the flow of fees which equals a fraction of the fund size (i.e., fP_b) is paid throughout the fund's lifetime. For calibration, we set $f \approx 2\%$ based on Metrick and Yasuda (2010), which finds that management fees are usually 2% of committed capital and paid from the inception of a fund until its liquidation.

First, we obtain the closed-form expression of PV_{dist}. Since the time to purchase, τ_b , is independent of the time to sell τ_s (post-purchase) and the selling price P_s,

$$PV_{\text{dist}} = E\left[e^{-r\tau_b}\right] E\left[\int_0^{\tau_s} e^{-rt}u(t)dt + e^{-r\tau_s}P_s\right],$$

where $u(t) \in \{u_f, u_e\}$ denotes the payoff flow while holding the asset at $t \in [0, \tau_s]$.

A purchase of an asset occurs on meeting a corporate investor or a fund at the exit phase, whichever happens first ($\tau_b \equiv \min\{\tau_{lo-fn}, \tau_{fe-fn}\}$). τ_b follows an exponential distribution with parameter $\lambda_f\mu_{lo} + \lambda_s\mu_{fe}$. As such,

$$E\left[e^{-r\tau_b}\right] = \frac{\lambda_f\mu_{lo} + \lambda_s\mu_{fe}}{\lambda_f\mu_{lo} + \lambda_s\mu_{fe} + r}.$$ [33]

The fund can sell either (i) before receiving a liquidity shock to a corporate investor, or (ii) after receiving a liquidity shock to either a corporate investor or a fund buyer. The

[33] We use (i) $\int_0^{\bar{t}} e^{-rt}dt = -\left.\frac{e^{-rt}}{r}\right|_0^{\bar{t}} = \frac{1-e^{-r\bar{t}}}{r}$, (ii) for $x \sim \exp(\alpha)$, $E[e^{-rx}] = \int_0^\infty e^{-rx}\alpha e^{-\alpha x}dx = \frac{\alpha}{\alpha+r}$, and (iii) for $x \sim \exp(\alpha)$, $E[\int_0^x e^{-rt}dt] = E\left[\frac{1-e^{-rx}}{r}\right] = \frac{1}{\alpha+r}$.

expected continuation payoff, upon receiving a liquidity shock before selling an asset, is

$$V_e \equiv E\left[u_e\left(\int_0^{\tau_e} e^{-rt}dt\right) + e^{-r\tau_e}P_e\right],$$

where τ_e denotes the time that the fund remains as type fe, and P_e denotes the selling price. Note that $\tau_e \equiv \min\{\tau_{fe-hn}, \tau_{fe-fn}\}$ follows an exponential distribution with parameter $\lambda_f\mu_{hn} + \lambda_s\mu_{fn}$. The probability of selling to a corporate investor $\frac{\lambda_f\mu_{hn}}{\lambda_f\mu_{hn}+\lambda_s\mu_{fn}}$ is independent of the selling time τ_e. Thus

$$\begin{aligned} V_e &= \frac{u_e}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn} + r} + \frac{\lambda_f\mu_{hn} + \lambda_s\mu_{fn}}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn} + r}\frac{\lambda_f\mu_{hn}P_{fe-hn} + \lambda_s\mu_{fn}P_{fe-fn}}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn}} \\ &= \frac{u_e + \lambda_f\mu_{hn}P_{fe-hn} + \lambda_s\mu_{fn}P_{fe-fn}}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn} + r}. \end{aligned}$$

Similarly, an fo type fund receives a payoff flow u_f during a lifetime spanning $\tau_{fo} \equiv \min\{\tau_{fo-hn}, \tau_e\}$. Eventually, the fund either sells its asset to a buying investor at price P_{fo-hn} or receives a liquidity shock and a continuation payoff V_e. Thus,

$$E\left[\int_0^{\tau_s} e^{-rt}u(t)dt + e^{-r\tau_s}P_s\right] = \frac{u_f + \lambda_f\mu_{hn}P_{fo-hn} + \rho_e V_e}{\lambda_f\mu_{hn} + \rho_e + r}.$$

It follows that

$$PV_{\text{dist}} = \left(\frac{\lambda_f\mu_{lo} + \lambda_s\mu_{fe}}{\lambda_f\mu_{lo} + \lambda_s\mu_{fe} + r}\right)\left(\frac{u_f + \lambda_f\mu_{hn}P_{fo-hn} + \rho_e\left(\frac{u_e+\lambda_f\mu_{hn}P_{fe-hn}+\lambda_s\mu_{fn}P_{fe-fn}}{\lambda_f\mu_{hn}+\lambda_s\mu_{fn}+r}\right)}{\lambda_f\mu_{hn} + \rho_e + r}\right).$$

Following a similar analysis (see the Supplemental Appendix) we derive the closed-form expression of PV_{calls}, which yields the following closed-form expression for PME.

Proposition 11. *(PME) The PE fund performance is given by*

$$PME = \frac{PV_{dist}}{PV_{calls}}, \tag{6}$$

where

$$PV_{dist} = \left(\frac{\lambda_f \mu_{lo} + \lambda_s \mu_{fe}}{\lambda_f \mu_{lo} + \lambda_s \mu_{fe} + r}\right) \left(\frac{u_f + \lambda_f \mu_{hn} P_{fo-hn} + \rho_e \left(\frac{u_e + \lambda_f \mu_{hn} P_{fe-hn} + \lambda_s \mu_{fn} P_{fe-fn}}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn} + r}\right)}{\lambda_f \mu_{hn} + \rho_e + r}\right), \textit{ and}$$

$$PV_{calls} = \frac{\lambda_f \mu_{lo} P_{lo-fn} + \lambda_s \mu_{fe} P_{fe-fn}}{\lambda_f \mu_{lo} + \lambda_s \mu_{fe} + r} \left(1 + f\left(\frac{1}{\lambda_f \mu_{lo} + \lambda_s \mu_{fe}} + \frac{1 + \rho_e \left(\frac{1}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn} + r}\right)}{\lambda_f \mu_{hn} + \rho_e + r}\right)\right).$$

We conclude this section by establishing various relationships among transaction prices (see proof in the Supplemental Appendix).

Proposition 12. *(Equilibrium Prices)*

1. $p_{fo-hn} \geq p_{fe-hn} \geq p_{fe-fn}$: *funds sell at a lower price during the exit phase than in the harvesting phase, and at an even lower price in secondary trading.*

2. $p_{fo-hn} \geq p_{lo-hn} \geq p_{lo-fn}$: *funds buy assets at a lower price and sell at a higher price than investors.*

Funds that manage to sell assets before receiving liquidity shocks can generate positive profits, from payoff flows u_f and the positive spread $p_{fo-hn} - p_{lo-fn}$. However, if funds suffer liquidity shocks before selling assets, they may incur losses ex-post because the spread at the exit phase $p_{fe-hn} - p_{lo-fn}$ can be negative, as is the case in our calibration. Further, the calibrated model predicts that doubling the number of funds from the 2012 level would result in a 0.8% decrease in the transaction price p_{lo-fn}.[34]

[34] In contrast, by studying venture capital data from 1987 to 1995, Gompers and Lerner (2000) report that doubling of venture capital funds would result in 15-35% increase in the price p_{lo-fn}. We believe that their finding is based on a partial equilibrium model, where increasing VCs that purchase assets would drive up the buying price p_{lo-fn}. However, this approach does not take into account the eventual need for VCs to sell assets.

Lastly, the sensitivity of prices and PME on flow payoffs are as follows:

$$\left[\frac{\partial x_i}{\partial \theta_j}\frac{\theta_j}{x_i}\right]_{i,j} = \left[\begin{array}{c|ccc} x_i \backslash \theta_j & u_h & u_f & u_e \\ \hline p_{lo-hn} & 0.379 & 0.019 & 0.021 \\ p_{lo-fn} & 0.337 & 0.043 & 0.043 \\ p_{fo-hn} & 0.370 & 0.043 & 0.044 \\ p_{fe-fn} & 0.329 & -0.043 & 0.087 \\ p_{fe-hn} & 0.372 & -0.019 & 0.065 \\ \text{PME} & -0.017 & 0.082 & 0.010 \end{array}\right].$$

Prices and PME change in intuitive directions in response to flow payoffs, but their elasticities are less than 1. For example, 1% increase in u_h would lead to transaction price increases for p_{lo-hn}, p_{lo-fn}, and p_{fo-hn} by 0.379%, 0.337%, and 0.370%, respectively. Also, an 1% increase in u_e is associated with an 0.087% increase p_{fe-fn}. An increase in u_f by 1% leads to a 0.082% increase in PME (Equation 6), thus confirming the role of operational improvements in fund returns.

7 Conclusion

We present a search-based model of asset trading with fund intermediation. The unique feature of the model is that funds operate under the risk of selling assets under pressure, potentially to other funds. We calibrate our model using data from PE buyout funds in the mid-size corporate acquisition market.

Our paper offers a novel explanation of persistent intermediators' returns when the number of intermediaries increases. An increase in the number of buyout funds can lead to a reduction in the potential opportunities for each fund to trade with investors in the primary market. However, it also offers increased opportunities to buy and sell assets in the secondary market. As a result, in aggregate, the ex-ante expected value of each fund at the beginning of its life cycle increases. This finding suggests that the benefits gained from participating in the secondary market outweigh the costs arising from the potentially more intense competition in the primary market. A well-lubricated private market for corporate acquisitions can partly explain the recent shift in firm ownership from public to private, predicated by Jensen (1991). Lower liquidity costs in a market for buying and selling private firms help make PE ownership a form of governance that may indeed eclipse public companies.

Appendix

A Mathematical Symbols

	Definition	**Notation**
	Number of investors	k_v
	Number of funds	k_f
	Number of assets	k_a
	Flow Payoff for low type investors	u_l
	Flow payoff for high type investors	u_h
	Flow payoff for funds in the harvesting phase	u_f
	Flow payoff for funds in the exiting phase	u_e
(Model parameters)	Rate of low valuation shock	ρ_d
	Rate of high valuation shock	ρ_u
	Rate of liquidity shock	ρ_e
	Match intensity (Direct trading)	λ_d
	Match intensity (Primary buyout)	λ_f
	Match intensity (Secondary buyout)	λ_s
	Discount rate	r
	(for each type $i \in \mathcal{T} \equiv \{hn, ln, ho, lo, fn, fo, fe\}$)	
	Population of type i agents	μ_i
	Value of type i agents	v_i
	Gains from trade between $i, j \in \mathcal{T}$	g_{i-j}
(Equilibrium variables)	Price of assets when $i, j \in \mathcal{T}$ trade	p_{i-j}
	Funds' Welfare	W_f
	Investors' welfare	W_v
	(Total) Welfare	W

Table 6: Mathematical Symbols

B Various Estimates of the Time to Sell

The model calibration determines parameter values that align with the data on PE activities. The data regarding the time to sell assets (Table 2) are sourced as follows. A sale of a private firm consists of two major processes: the preparation and the listing-to-sale process. The preparation takes less time if a firm already has high-quality accounting and information systems, which is the case of PE-backed firms (Kaplan and Stromberg (2009)). The preparation for PE-backed firms takes an average of 2 months, while other firms need an average of 6 months (see the upper part of Table 7). The listing-to-sale process takes about 9 months for various selling agents (see the lower part of Table 7). We set the total time for selling a firm as 11 months for PE funds and 15 months for corporate investors.

Ave. Time Taken	Source
For preparations	
1-6 months	https://www.highrockpartners.com/how-long-does-it-take-to-sell-a-company/
12 months	https://www.businessinsider.com/11-stages-of-selling-a-company-2011-4
From listing to sale	
6-9 months	https://www.mabusinessbrokers.com/blog/how-long-does-it-take-to-sell-a-business
9 months	https://www.exitadviser.com/seller-status.aspx?id=long-does-take-sell
9 months	https://www.allbusiness.com/how-long-does-it-take-to-sell-a-business-2-6592268-1.html
12 months	https://www.businessinsider.com/11-stages-of-selling-a-company-2011-4
9 months	https://www.moorestephens.co.uk/msuk/moore-stephens-south/news/july-2017-(1) /how-long-does-it-take-to-sell-a-small-business
9 months	https://www.tvba.co.uk/article/how-long-does-it-take-to-sell-a-company
6-9 months	https://www.simonscottcmc.co.uk/blog/long-take-sell-business/
10 months	https://www.ibgbusiness.com/tips-sell-business-long-take-sell-business/
10 months	https://www.highrockpartners.com/how-long-does-it-take-to-sell-a-company/

Table 7: Estimated time to sell a firm

C Proofs

C.1 Proof for Part 1 of Proposition 1

We reduce the number of variables and population equations in $P(\theta)$ by imposing some necessary conditions for a steady-state solution. Note that any steady-state solution μ must satisfy $\mu_{ho} + \mu_{hn} = k_h \equiv \frac{\rho_u}{\rho_u+\rho_d}$ and $\mu_{lo} + \mu_{ln} = k_l \equiv \frac{\rho_d}{\rho_u+\rho_d}$ (which we can obtain by adding (μ-ho) and (μ-hn), or (μ-lo) and (μ-ln), and apply $k_v = 1$). If we substitute $\mu_{ho} = k_h - \mu_{hn}$ and $\mu_{ln} = n_l - \mu_{lo}$ into (μ-ho)-(μ-fe), then we are left with the following three linearly independent equations:[35]

$$
\begin{aligned}
(\lambda_d \mu_{hn} + \lambda_f \mu_{fn})\mu_{lo} + \rho_u \mu_{lo} - \rho_d \mu_{ho} &= 0, && \text{(from (μ-lo))} \\
(\lambda_d \mu_{lo} + \lambda_f \mu_{fo} + \lambda_f \mu_{fe})\mu_{hn} + \rho_d \mu_{hn} - \rho_u \mu_{ln} &= 0, && \text{(from (μ-hn))} \\
-(\lambda_f \mu_{hn} + \lambda_s \mu_{fn})\mu_{fe} + \rho_e \mu_{fo} &= 0. && \text{(from (μ-fe))}
\end{aligned}
$$

We re-write the first two equations with respect to μ_{lo} and μ_{hn}.

$$\mu_{fo} + \mu_{fe} = k_a - \mu_{ho} - \mu_{lo} = k_a - (k_h - \mu_{hn}) - \mu_{lo} \quad \text{and} \tag{7}$$

$$\mu_{fn} = k_f - (\mu_{fo} + \mu_{fe}) = k_f - k_a + k_h - \mu_{hn} + \mu_{lo}, \tag{8}$$

Then,

$$(\lambda_d \mu_{hn} + \lambda_f (k_f - k_a + k_h - \mu_{hn} + \mu_{lo}))\mu_{lo} + \rho_u \mu_{lo} - \rho_d (k_h - \mu_{hn}) = 0, \tag{9}$$

$$(\lambda_d \mu_{lo} + \lambda_f (k_a - k_h + \mu_{hn} - \mu_{lo}))\mu_{hn} + \rho_d \mu_{hn} - \rho_u (k_l - \mu_{lo}) = 0. \tag{10}$$

We show below that there exists a unique solution (μ_{lo}, μ_{hn}) of (9)-(10) such that (i) $0 \le \mu_{lo} \le k_l$, (ii) $0 \le \mu_{hn} \le k_h$, and (iii) $k_a - k_f - k_h \le \mu_{lo} - \mu_{hn} \le k_a - k_h$ (for $0 \le \mu_{fn} \le k_f$). Other population measures will be determined by $\mu_{ho} = k_h - \mu_{hn}$, $\mu_{ln} = k_l - \mu_{lo}$, and

$$\mu_{fn} = k_f - k_a + k_h - \mu_{hn} + \mu_{lo}. \tag{11}$$

We find the last two populations (μ_{fe}, μ_{fo}) by solving

$$
\begin{aligned}
\mu_{fo} &= -\mu_{fe} + (k_f - \mu_{fn}), \quad \text{(from $\mu_{fn} + \mu_{fo} + \mu_{fe} = k_f$)} \\
\mu_{fo} &= \frac{\lambda_f \mu_{hn} + \lambda_s \mu_{fn}}{\rho_e} \mu_{fe}. \quad \text{(from (μ-fe))}
\end{aligned}
$$

[35] Any other equation in $P(\theta)$ is redundant, as it depends linearly on (μ-lo), (μ-hn), and (μ-fe). Each sum of the right-hand sides of (μ-ho) and (μ-hn), or (μ-lo) and (μ-ln) equals zero, which allows us to delete (μ-ho) and (μ-ln) without changing the solution set. The sum of the right-hand sides of (μ-fn), (μ-fo), and (μ-fe) equals zero, so we can delete (μ-fn). Last, the sum of the right-hand sides of (μ-ho), (μ-lo), (μ-fo), and (μ-fe) equals zero, so we can delete (μ-fo).

The unique solution is

$$\mu_{fe} = \frac{\rho_e(k_f - \mu_{fn})}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn} + \rho_e}, \quad \text{and} \quad \mu_{fo} = \frac{(\lambda_f\mu_{hn} + \lambda_s\mu_{fn})(n_f - \mu_{fn})}{\lambda_f\mu_{hn} + \lambda_s\mu_{fn} + \rho_e}. \tag{12}$$

Therefore, it remains to prove the following claim:

Claim 1. *Let* $X(\theta) \equiv \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le k_l, 0 \le x_2 \le k_h, 0 \le g_{fn}(x) \le k_f\}$, *where* $g_{fn}(x) \equiv k_a - k_h + x_2 - x_1$. *Also, define* $F \equiv (F_{lo}, F_{hn}) : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$F_{lo}(x) \equiv (\lambda_d x_2 + \lambda_f(k_f - g_{fn}(x)))x_1 + \rho_u x_1 - \rho_d(k_h - x_2),$$
$$F_{hn}(x) \equiv (\lambda_d x_1 + \lambda_f g_{fn}(x))x_2 + \rho_d x_2 - \rho_u(k_l - x_1).$$

Then, there exists a unique solution of $F(x) = 0$ *in* $X(\theta)$.

We apply the Poincare-Hopf index theorem, a version in Simsek et al. (2007, p.194); see also Hirsch (2012). First, $X(\theta)$ is non-empty, compact, and convex.[36] The boundary of $X(\theta)$ is

$$\partial X(\theta) \equiv \{(x_1, x_2) \in X(\theta) : x_1 = 0, x_1 = k_l, x_2 = 0, x_2 = k_h, g_{fn}(x) = 0, \text{ or } g_{fn}(x) = k_f\}.$$

Second, the function $F(x)$ is continuously differentiable at every $x \in \mathbb{R}^2$. Third, the determinant of the Jacobian matrix of F is strictly positive for every interior point of $X(\theta)$: for each $x \in \mathbb{R}^2$,

$$\begin{aligned}\nabla F(x) &\equiv \begin{bmatrix} \frac{\partial F_{lo}}{\partial x_1} & \frac{\partial F_{lo}}{\partial x_2} \\ \frac{\partial F_{hn}}{\partial x_1} & \frac{\partial F_{hn}}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} (\lambda_d x_2 + \lambda_f(k_f - g_{fn}(x))) + \lambda_f x_1 + \rho_u & (\lambda_d - \lambda_f)x_1 + \rho_d \\ (\lambda_d - \lambda_f)x_2 + \rho_u & (\lambda_d x_1 + \lambda_f g_{fn}(x)) + \lambda_f x_2 + \rho_d \end{bmatrix},\end{aligned}$$

[36] The Poincare-Hopf index theorem also requires $X(\theta)$ to be a 2-dimensional smooth manifold, which a reader can easily verify by applying the identify function to the definition of a smooth manifold in Simsek et al. (2007, p.193).

and for any interior point $x \in X(\theta)\backslash\partial X(\theta)$,

$$\begin{aligned} det(\nabla F(x)) \geq& (\lambda_d x_2 + \lambda_f x_1)(\lambda_d x_1 + \lambda_f x_2) + \rho_u \lambda_d x_1 + \rho_d \lambda_d x_2 \\ & - (\lambda_d - \lambda_f)^2 x_1 x_2 - (\lambda_d - \lambda_f)(\rho_d x_2 + \rho_u x_1) \\ =& \lambda_d \lambda_f (x_1^2 + x_2^2) + 2\lambda_d \lambda_f x_1 x_2 + \lambda_f(\rho_d x_2 + \rho_u x_1) > 0. \end{aligned} \tag{13}$$

Last, we show that for every boundary point $x \in \partial X(\theta)$, the vector $F(x) \in \mathbb{R}^2$ points strictly outward of $X(\theta)$. We partition the boundary $\partial X(\theta)$ into six *faces* (i.e., flat surfaces) of $X(\theta)$. For each face, we find an outward normal vector $\mathbf{n} \in \mathbb{R}^2$ and show that the angle between $\mathbf{n}$ and $F(x)$ is acute (i.e., ≤ 90) at any point x in the face:

1. ($x_1 = 0$ and $0 \leq x_2 < k_h$) $\mathbf{n} = (-1, 0)$ is an outward normal vector, and $\mathbf{n} \cdot F(x) = \rho_d(k_h - x_2) > 0$.

2. ($x_2 = 0$ and $0 \leq x_1 < k_l$) $\mathbf{n} = (0, -1)$ is an outward normal vector, and $\mathbf{n} \cdot F(x) = \rho_u(k_l - x_1) > 0$.

3. ($x_1 = k_l$ and $0 \leq x_2 \leq k_h$) $\mathbf{n} = (1, 0)$ is an outward normal vector, and

$$\begin{aligned} \mathbf{n} \cdot F(x) &= (\lambda_d x_2 + \lambda_f(k_f - g_{fn}(x)))k_l + \rho_u k_l - \rho_d k_h + \rho_d x_2 \\ &\geq (\lambda_d x_2 + \lambda_f(k_f - g_{fn}(x)))k_l \quad (\text{as } \rho_u k_l = \rho_d k_h) \\ &\geq \min\{\lambda_d x_2 k_l, \lambda_f(k_f - g_{fn}(x))k_l\}. \end{aligned}$$

   As either $x_2 > 0$ or $x_2 = 0$, we have $k_f - g_{fn}(x) = k_v + k_f - k_a > 0$, and $\mathbf{n} \cdot F(x) > 0$.

4. ($x_2 = k_h$ and $0 \leq x_1 \leq k_l$) $\mathbf{n} = (0, 1)$ is an outward normal vector, and

$$\mathbf{n} \cdot F(x) = (\lambda_d x_1 + \lambda_f g_{fn}(x))k_h + \rho_d k_h - \rho_u k_l + \rho_u x_1 \geq \min\{\lambda_d x_1 k_h, \lambda_f g_{fn}(x) k_h\}.$$

   As either $x_1 > 0$ or $x_1 = 0$, we have $g_{fn}(x) = k_a > 0$, and $\mathbf{n} \cdot F(x) > 0$.

5. ($g_{fn}(x) = 0$ and $x_1 > 0$) $\mathbf{n} = (1, -1)$ is an outward normal vector, and $\mathbf{n} \cdot F(x) = F_{lo}(x) - F_{hn}(x) = \lambda_f k_f x_1 > 0$.

6. ($g_{fn}(x) = k_f$ and $x_2 > 0$) $\mathbf{n} = (-1, 1)$ is an outward normal vector, and $\mathbf{n} \cdot F(x) = F_{hn}(x) - F_{lo}(x) = \lambda_f k_f x_2 > 0$.

We are ready to apply the Poincare-Hopf index theorem in Simsek et al. (2007, p.194). The Euler characteristic of $X(\theta)$ is 1; see their definition on p.193 for the case of non-empty and convex sets. Claim 1 follows immediately from the index theorem, which completes the proof of existence and uniqueness of the solution of $P(\theta)$.

It remains to prove:

Lemma 1. *If μ is a steady-state solution of $P(\theta)$, then $\mu_i > 0$ for all $i \in \mathcal{T}$.*

The intuition of the lemma is simple. Strictly positive rates $\lambda = (\lambda_d, \lambda_f, \lambda_s)$ and $\rho = (\rho_u, \rho_d, \rho_e)$ allow the mass of investors and funds to flow across all types. Given $0 < k_a < k_v + k_f$, some fraction of agents are owners and others are non-owners. Investors who own assets may have their types changing between high and low exogenously and non-owners have similar probabilistic type changes. As transaction rates $\lambda = (\lambda_d, \lambda_f, \lambda_s)$ are all strictly positive, some investors can buy or offload assets after their types change. However, not all investors can do so within any fixed time period. A similar idea holds for funds.

Proof. First, we show that $\mu_{ho} > 0$ and $\mu_{lo} > 0$. It is clear that $\mu_{ho} = 0$ if and only if $\mu_{lo} = 0$. If $\mu_{ho} = 0$, then $\mu_{lo} = 0$, as only *ho*-type investors flow in type *lo*; conversely, if $\mu_{lo} = 0$, then $\mu_{ho} = 0$ as the inflow to the type *lo* must be zero. Suppose, toward contradiction, that $\mu_{ho} = \mu_{lo} = 0$. That is, all investors are non-owners. The in-flow from *hn*-type investors to type *ho* must be zero, so it must be that $\mu_{fo} = \mu_{fe} = 0$. Then, $\mu_{ho} + \mu_{lo} + \mu_{fo} + \mu_{fe} = 0$, a contradiction to $k_a > 0$.

Second, we show that $\mu_{ln} > 0$ and $\mu_{hn} > 0$. As before, it is clear that $\mu_{ln} = 0$ if and only if $\mu_{hn} = 0$. If $\mu_{ln} = 0$, then $\mu_{hn} = 0$ as only *ln*-type investors can flow in type *hn*; conversely, if $\mu_{hn} = 0$, then $\mu_{ln} = 0$ as the inflow to the type *hn* must be zero. Suppose, toward contradiction, that $\mu_{ln} = \mu_{hn} = 0$. That is, all investors are owners, which implies that some funds are non-owners: $\mu_{fn} = k_v + k_f - k_a > 0$. Since $\lambda_f \mu_{lo} \mu_{fn} > 0$, some *lo*-type investors change their types and flow into type *ln* by trading with PE funds, a contradiction to $\mu_{ln} = 0$.

Lastly, we consider funds. Suppose that $\mu_{fn} = 0$. As the inflow to type fo becomes zero, it must be that $\mu_{fo} = 0$, which in turn leads to no inflow by liquidity shocks to type fe: i.e., $\mu_{fe} = 0$. Such a case contradicts $k_f > 0$. When $\mu_{fn} > 0$, given strictly positive population μ_{lo}, the inflow of type-fn funds to type fo is strictly positive: $\lambda_f \mu_{lo} \mu_{fn} > 0$. As such, $\mu_{fo} > 0$, which in turn creates a strictly positive inflow by liquidity shocks to type fe: $\mu_{fe} > 0$. □

C.2 Proof for Part 2 of Proposition 1

We first reduce the system $P(\theta)$. For any initial condition $\mu(0)$, a dynamic solution $\mu : [0, \infty) \to \mathbb{R}^{\mathcal{T}}$ of the system $P(\theta)$ satisfies, for every $t \in [0, \infty)$,

$$\begin{aligned} \mu_{ho}(t) + \mu_{hn}(t) + \mu_{lo}(t) + \mu_{ln}(t) &= k_v (= 1), \\ \mu_{fn}(t) + \mu_{fo}(t) + \mu_{fe}(t) &= k_f, \quad \text{and} \\ \mu_{ho}(t) + \mu_{lo}(t) + \mu_{fo}(t) + \mu_{fe}(t) &= k_a. \end{aligned}$$

Without changing the set of dynamic solutions, we can reduce the system $P(\theta)$ for $x(t) \equiv (\mu_{ho}(t), \mu_{hn}(t), \mu_{lo}(t), \mu_{fo}(t))$ by[37]

$$\dot{x} = F(x) \equiv \left(F_{ho}\left(x\right), F_{hn}\left(x\right), F_{lo}\left(x\right), F_{fo}\left(x\right)\right), \tag{14}$$

[37]In the proof of Part 1 of Proposition 1, we reduced $F(x;\theta) = 0$ further as a system of only two equations in Claim 1. The reduction requires $\mu_{hn} + \mu_{ho} = k_h \equiv \frac{\rho_u}{\rho_u + \rho_d}$ and $\mu_{lo} + \mu_{ln} = k_l \equiv \frac{\rho_d}{\rho_u + \rho_d}$, which hold in a steady-state but may not hold on a path of $\mu(t)$ after a perturbation.

where

$$
\begin{aligned}
F_{ho}(x) &\equiv (\lambda_d\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe}(x))\mu_{hn} - \rho_d\mu_{ho} + \rho_u\mu_{lo}, \\
F_{hn}(x) &\equiv -(\lambda_d\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe}(x))\mu_{hn} - \rho_d\mu_{hn} + \rho_u\mu_{ln}(x), \\
F_{lo}(x) &\equiv -(\lambda_d\mu_{hn} + \lambda_f\mu_{fn}(x))\mu_{lo} - \rho_u\mu_{lo} + \rho_d\mu_{ho}, \\
F_{fo}(x) &\equiv (\lambda_f\mu_{lo} + \lambda_s\mu_{fe}(x))\,\mu_{fn}(x) - \lambda_f\mu_{hn}\mu_{fo} - \rho_e\mu_{fo}, \quad \text{and} \\
\mu_{ln}(x) &\equiv 1 - \mu_{ho} - \mu_{hn} - \mu_{lo}, \\
\mu_{fe}(x) &\equiv k_a - \mu_{ho} - \mu_{lo} - \mu_{fo}, \\
\mu_{fn}(x) &\equiv k_f - \mu_{fo} - \mu_{fe}(x) = k_f - k_a + \mu_{ho} + \mu_{lo}.
\end{aligned}
\tag{15}
$$

The reduction of the system $P(\theta)$ does not change the set of dynamic solutions. If μ is a dynamic (either steady-state or not) solution of $P(\theta)$, then $x \equiv (\mu_{ho}, \mu_{hn}, \mu_{lo}, \mu_{fo})$ solves $F(x;\theta) = 0$; conversely, for any dynamic solution x of $F(x;\theta) = 0$, we can find a dynamic solution μ of $P(\theta)$, from x and the induced μ_{ln}, μ_{fe}, and μ_{fn}. Hence, a dynamic solution μ of $P(\theta)$ is asymptotically stable if and only if $x \equiv (\mu_{ho}, \mu_{hn}, \mu_{lo}, \mu_{fo})$ is asymptotically stable.

A steady-state solution x of $F(x;\theta) = 0$ is asymptotically stable if all eigenvalues of the Jacobian matrix of $F(x;\theta)$ at the steady-state solution x have strictly negative real parts (Hirsch, 2012). The Jacobian matrix is

$$
\nabla F(x) \equiv \left[\frac{\partial F_i(x)}{\partial x_j}\right]_{i,j\in\{ho,hn,lo,fo\}}
$$
$$
= \left[\begin{array}{ccc|c}
-\lambda_f\mu_{hn} - \rho_d & \lambda_v\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe} & (\lambda_d - \lambda_f)\mu_{hn} + \rho_u & 0 \\
\lambda_f\mu_{hn} - \rho_u & -\lambda_v\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe}) - \rho_d - \rho_u & (\lambda_f - \lambda_d)\mu_{hn} - \rho_u & 0 \\
-\lambda_f\mu_{lo} + \rho_d & -\lambda_d\mu_{lo} & -\lambda_f(\mu_{fn} + \mu_{lo}) - \lambda_d\mu_{hn} - \rho_u & 0 \\
\hline
\lambda_f\mu_{lo} + \lambda_s(\mu_{fe} - \mu_{fn}) & -\lambda_f\mu_{fo} & \lambda_f(\mu_{fn} + \mu_{lo}) + \lambda_s(\mu_{fe} - \mu_{fn}) & -\lambda_f\mu_{hn} - \lambda_s\mu_{fn} - \rho_e
\end{array}\right]
$$

where we omit the dependency of μ_{fn} and μ_{fe} on x to simplify the expression.

Due to the block structure, one eigenvalue is $-\lambda_f\mu_{hn} - \lambda_s\mu_{fn} - \rho_e < 0$. The other eigenvalues are the eigenvalues of the sub-matrix with the first three rows and columns. A direct calculation shows that the other three eigenvalues are also strictly negative, which completes the proof.

C.3 Proof of Proposition 2

First, we simplify expositions:

$$
\begin{aligned}
g_1 &\equiv g_{fo-hn} = (1/2)(v_{ho} + v_{fn} - v_{fo} - v_{hn}),\\
g_2 &\equiv g_{lo-fn} = (1/2)(v_{fo} + v_{ln} - v_{lo} - v_{fn}),\\
g_3 &\equiv g_{fe-fn} = (1/2)(v_{fo} + v_{fn} - v_{fe} - v_{fn}) = (1/2)(v_{fo} - v_{fe}),
\end{aligned}
$$

so that

$$
\begin{aligned}
g_{lo-hn} &= (1/2)(v_{ho} + v_{ln} - v_{lo} - v_{hn}) = g_2 + g_1 \quad \text{and}\\
g_{fe-hn} &= (1/2)(v_{ho} + v_{fn} - v_{fe} - v_{hn}) = g_1 + g_3.
\end{aligned}
$$

The matrix representations of the value equations (v-hn)-(v-fe) are:

$$
\begin{aligned}
\begin{bmatrix} v_{ho} \\ v_{lo} \end{bmatrix} &= \begin{bmatrix} r+\rho_d & -\rho_d \\ -\rho_u & r+\rho_u \end{bmatrix}^{-1} \begin{bmatrix} u_h \\ u_l + \lambda_d \mu_{hn}(g_1+g_2) + \lambda_f \mu_{fn} g_2 \end{bmatrix},\\
\begin{bmatrix} v_{hn} \\ v_{ln} \end{bmatrix} &= \begin{bmatrix} r+\rho_d & -\rho_d \\ -\rho_u & r+\rho_u \end{bmatrix}^{-1} \begin{bmatrix} \lambda_d \mu_{lo}(g_1+g_2) + \lambda_f \mu_{fo} g_1 + \lambda_f \mu_{fe}(g_1+g_3) \\ 0 \end{bmatrix}, \quad \text{and}\\
\begin{bmatrix} v_{fn} \\ v_{fo} \\ v_{fe} \end{bmatrix} &= \frac{1}{r} \begin{bmatrix} \lambda_f \mu_{lo} g_2 + \lambda_s \mu_{fe} g_3 \\ u_f + \lambda_f \mu_{hn} g_1 - 2\rho_e g_3 \\ u_e + \lambda_f \mu_{hn}(g_1+g_3) + \lambda_s \mu_{fn} g_3 \end{bmatrix},
\end{aligned} \tag{16}
$$

where the inverse matrix is well-defined: i.e., $(r+\rho_d)(r+\rho_u) - \rho_d\rho_u > 0$. As in the case of $k_f = 0$, we compute the gains g_1, g_2, and g_3. Then, the solution v will be uniquely determined by the above matrix equations.

First, $2rg_3 = r(v_{fo} - v_{fe}) = (u_f - u_e) - 2\rho_e g_3 - \lambda_f \mu_{hn} g_3 - \lambda_s \mu_{fn} g_3$, which implies

$$
g_3 = \frac{u_f - u_e}{2r + 2\rho_e + \lambda_f \mu_{hn} + \lambda_s \mu_{fn}} > 0. \tag{17}
$$

Next,

$$\begin{aligned}2(g_1+g_2) &= v_{ho}+v_{ln}-v_{lo}-v_{hn} = (1,-1)\cdot(v_{ho}-v_{hn}, v_{lo}-v_{ln}) \\ &= \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}r+\rho_d & -\rho_d \\ -\rho_u & r+\rho_u\end{bmatrix}^{-1}\begin{bmatrix}u_h-\lambda_d\mu_{lo}(g_1+g_2)-\lambda_f\mu_{fo}g_1-\lambda_f\mu_{fe}(g_1+g_3) \\ u_l+\lambda_d\mu_{hn}(g_1+g_2)+\lambda_f\mu_{fn}g_2\end{bmatrix}.\end{aligned}$$

Since

$$\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}r+\rho_d & -\rho_d \\ -\rho_u & r+\rho_u\end{bmatrix}^{-1} = \frac{1}{r+\rho_u+\rho_d}\begin{bmatrix}1 & -1\end{bmatrix},$$

we have

$$\begin{aligned}&(2(r+\rho_u+\rho_d)+\lambda_d(\mu_{lo}+\mu_{hn}))(g_1+g_2)+\lambda_f(\mu_{fo}+\mu_{fe})g_1+\lambda_f\mu_{fn}g_2 \\ &\qquad = (u_h-u_l)-\lambda_f\mu_{fe}g_3. \qquad (18)\end{aligned}$$

On the other hand, by (v-lo), (v-ln), (v-fn), and (v-fo),

$$\begin{aligned}2rg_2 =& r(v_{fo}-v_{fn})-r(v_{lo}-v_{ln}) \\ =& (u_f+\lambda_f\mu_{hn}g_1-2\rho_e g_3-\lambda_f\mu_{lo}g_2-\lambda_s\mu_{fe}g_3) \\ &-(u_l+\lambda_d\mu_{hn}(g_1+g_2)+\lambda_f\mu_{fn}g_2)+\rho_u(v_{ho}-v_{lo}+v_{ln}-v_{hn}).\end{aligned}$$

As $v_{ho}-v_{lo}+v_{ln}-v_{hn}=2(g_1+g_2)$,

$$\begin{aligned}&(2\rho_u+\lambda_d\mu_{hn})(g_1+g_2)-\lambda_f\mu_{hn}g_1+(2r+\lambda_f\mu_{lo}+\lambda_f\mu_{fn})g_2 \\ &\qquad = (u_f-u_l)-(2\rho_e+\lambda_s\mu_{fe})g_3. \qquad (19)\end{aligned}$$

The linear system of equations (18) and (19) is summarized as follows:

$$\begin{bmatrix}c_1+\lambda_f(\mu_{fo}+\mu_{fe}) & c_1+\lambda_f\mu_{fn} \\ c_2-\lambda_f\mu_{hn} & c_2+2r+\lambda_f(\mu_{lo}+\mu_{fn})\end{bmatrix}\begin{bmatrix}g_1 \\ g_2\end{bmatrix} = \begin{bmatrix}u_h-u_l-\lambda_f\mu_{fe}g_3 \\ u_f-u_l-2\rho_e g_3-\lambda_s\mu_{fe}g_3\end{bmatrix} \qquad (20)$$

where $c_1 \equiv 2(r+\rho_u+\rho_d)+\lambda_d(\mu_{lo}+\mu_{hn})>0$ and $c_2 \equiv 2\rho_u+\lambda_d\mu_{hn}>0$.

The determinant of the coefficient matrix is bounded below by

$$2rc_1 + \lambda_f \mu_{fn}(c_1 - c_2) > 4r^2 + \lambda_f \mu_{fn}(2r + 2\rho_d + \lambda_d \mu_{lo}) > 0.$$

Thus, the above linear system has a unique solution (g_1, g_2). This solution, together with g_3, determined the unique solution v of $V(\theta)$. ■

C.4 Conditions for positive trade gains

The gains from trade are all positive if and only if

$$\begin{aligned} g_1 \equiv g_{fo-hn} \geq 0 \iff & (c_2 + 2r + \lambda_f(\mu_{lo} + \mu_{fn}))((u_h - u_l) - \lambda_f \mu_{fe} g_3) \\ & - (c_1 + \lambda_f \mu_{fn})((u_f - u_l) - (2\rho_e + \lambda_s \mu_{fe})g_3) \geq 0. \quad (21) \\ g_2 \equiv g_{lo-fn} \geq 0 \iff & -(c_2 - \lambda_f \mu_{hn})((u_h - u_l) - \lambda_f \mu_{fe} g_3) \\ & + (c_1 + \lambda_f(\mu_{fo} + \mu_{fe}))((u_f - u_l) - (2\rho_e + \lambda_s \mu_{fe})g_3) \geq 0, \quad (22) \end{aligned}$$

Note that both expressions depend on the steady-state population measure μ.

C.5 Proof of Proposition 3

C.5.1 Part 1

Let $(\mu(\theta), v(\theta))$ be the unique steady-state solution of population and value for each market θ. We compute the comparative static derivatives with respect to λ_s. It is intuitive that the unique steady-state measure of each investor type $(\mu_i)_{i \in \mathcal{T}_v}$ and the measure μ_{fn} are independent of λ_s. Through a secondary trade, one fund changes its type from fe to fn, replacing another fund of type changed from fn to fo.

To confirm the intuition, from the proof of Proposition 1, take the unique steady-state solution $x(\theta) \equiv (\mu_{lo}(\theta), \mu_{hn}(\theta))$ of $F(x) \equiv (F_{lo}(x), F_{hn}(x)) = 0$, where

$$\begin{aligned} F_{lo}(x) &\equiv (\lambda_d x_2 + \lambda_f(k_f - g_{fn}(x)))x_1 + \rho_u x_1 - \rho_d(k_h - x_2), \\ F_{hn}(x) &\equiv (\lambda_d x_1 + \lambda_f g_{fn}(x))x_2 + \rho_d x_2 - \rho_u(k_l - x_1). \end{aligned}$$

By Implicit function theorem, $x(\theta)$ is differentiable in λ_s, and

$$\frac{\partial x(\theta)}{\partial \lambda_s} = -\left[\nabla_x F(x(\theta);\theta)\right]^{-1} \frac{\partial F(x(\theta);\theta)}{\partial \lambda_s}.$$

We denoted the domain of $F(x)$ by $X(\theta)$ in the proof of Claim 1. The unique solution $x(\theta)$ of $F(x;\theta) = 0$ is an interior point of $X(\theta)$, as shown in Lemma 1 for the case of $k_f > 0$ and in the proof of Part 1 of Proposition 1 for the case of $k_f = 0$. As a result, we have shown in the proof of Claim 1, the Jacobian matrix $\nabla F(x)$ at the unique solution $x(\theta)$ is invertible.

Then,

$$\frac{\partial F(x(\theta);\theta)}{\partial \lambda_s} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \frac{\partial x(\theta)}{\partial \lambda_s} = \begin{bmatrix} \partial \mu_{lo}(\theta)/\partial \lambda_s \\ \partial \mu_{hn}(\theta)/\partial \lambda_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{23}$$

For other types, it follows from $\mu_{ho}(\theta) + \mu_{hn}(\theta) = k_h$ and $\mu_{lo}(\theta) + \mu_{ln}(\theta) = k_l$ that $\frac{\partial \mu_{ln}(\theta)}{\partial \lambda_s} = \frac{\partial \mu_{ho}(\theta)}{\partial \lambda_s} = 0$, and from $\mu_{ho}(\theta) + \mu_{lo}(\theta) + (k_f - \mu_{fn}(\theta)) = k_f$ that $\frac{\partial \mu_{fn}(\theta)}{\partial \lambda_s} = 0$. Lastly, from (12) and $\mu_{fo} + \mu_{fe} + \mu_{fn} = k_f$,

$$\frac{\partial \mu_{fe}(\theta)}{\partial \lambda_s} = \frac{-\rho_e(k_f - \mu_{fn}(\theta))\mu_{fn}(\theta)}{(\rho_e + \lambda_f \mu_{hn}(\theta) + \lambda_s \mu_{fn}(\theta))^2} = -\frac{\partial \mu_{fo}(\theta)}{\partial \lambda_s}. \tag{24}$$

From the definition of g_3 on p. 44 and Equation 17,

$$v_{fo} - v_{fe} = 2g_{fe-fn} = 2g_3 = \frac{2(u_f - u_e)}{2r + 2\rho_e + \lambda_f \mu_{hn} + \lambda_s \mu_{fn}}.$$

The comparative static derivatives show that μ_{hn} and μ_{fn} are independent of λ_s. Thus, $\frac{\partial(v_{fo} - v_{fe})}{\partial \lambda_s} < 0$ and $\lim_{\lambda_s \to \infty}(v_{fo} - v_{fe}) = 0$.

C.5.2 Part 2

The above comparative static derivatives with respect to λ_s show that $\frac{\partial \mu_{fo}(\theta)}{\partial \lambda_s} > 0$, $\frac{\partial \mu_{fe}(\theta)}{\partial \lambda_s} < 0$, and $\frac{\partial \mu_i(\theta)}{\partial \lambda_s} = 0$, for all $i \neq fo, fe$. Thus,

$$\begin{aligned} r\frac{\partial W(\theta)}{\partial \lambda_s} &= \frac{\partial \mu_{ho}(\theta)}{\partial \lambda_s}u_h + \frac{\partial \mu_{lo}(\theta)}{\partial \lambda_s}u_l + \frac{\partial \mu_{fo}(\theta)}{\partial \lambda_s}u_f + \frac{\partial \mu_{fe}(\theta)}{\partial \lambda_s}u_e \\ &= \frac{\partial \mu_{fo}(\theta)}{\partial \lambda_s}(u_f - u_e) + \frac{\partial(\mu_{fo} + \mu_{fe})(\theta)}{\partial \lambda_s}u_e \\ &= \frac{\partial \mu_{fo}(\theta)}{\partial \lambda_s}(u_f - u_e) - \frac{\partial \mu_{fn}(\theta)}{\partial \lambda_s}u_e = \frac{\partial \mu_{fo}(\theta)}{\partial \lambda_s}(u_f - u_e) > 0. \end{aligned}$$

C.6 Proof of Proposition 4 and Proposition 5

Recall from Claim 1 that the steady-state population is determined by a solution of $F(x; k_f) \equiv (F_{lo}(x; k_f), F_{hn}(x; k_f)) = 0$ where

$$\begin{aligned} F_{lo}(x; k_f) &\equiv (\lambda_d x_2 + \lambda_f(k_f - g_{fn}(x)))x_1 + \rho_u x_1 - \rho_d(k_h - x_2), \\ F_{hn}(x; k_f) &\equiv (\lambda_d x_1 + \lambda_f g_{fn}(x))x_2 + \rho_d x_2 - \rho_u(k_l - x_1), \end{aligned}$$

and $g_{fn}(x) \equiv k_a - k_h + x_2 - x_1$. We extend the system $F(x; k_f) = 0$ such that k_f can be any real number and x can be any real vector of length 2. Each solution $x = (x_1, x_2)$ defines a vector $\mu = (\mu_i)_{i \in \mathcal{T}}$ as $(\mu_{lo}, \mu_{hn}, \mu_{ln}, \mu_{ho}) = (x_1, x_2, k_l - x_1, k_h - x_2)$ and $(\mu_{fn}, \mu_{fo}, \mu_{fe})$ by (11) and (12). According to Claim 1, if $k_f > 0$, a solution exists in certain domain (denoted by $X(\theta)$ in the claim) such that the resulting vector μ is a steady-state population. In general, without any restrictions on k_f, the vector μ may not even be positive.

The proof consists of three steps. First, for $k_f = 0$, we find a population measure $\hat{\mu}$ such that $\hat{x} \equiv (\hat{\mu}_{lo}, \hat{\mu}_{hn})$ solves $F(x; k_f) = 0$. Second, by Implicit Function Theorem, we differentiate a solution function $x(k_f)$ defined in the neighborhood of $k_f = 0$ and $x = \hat{x}$, and obtain the comparative static derivative $\mu_i' \equiv \left.\frac{\partial \mu_i}{\partial k_f}\right|_{k_f=0}$ for each $i \in \mathcal{T}$. Last, we prove the following claim:

Claim 2. *There exist $\beta_1 > 0$ and β_2, each being independent of λ_s, such that*

$$\left.\frac{\partial v_{fn}}{\partial k_f}\right|_{k_f=0} = \beta_1 \lambda_s + \beta_2.$$

Then, Proposition 4 and Proposition 5 follow immediately.

C.6.1 A benchmark model ($k_f = 0$)

We set $\hat{\mu}_i = 0$ for every fund type $i \in \mathcal{T}_f$, and impose

$$\begin{aligned}
&\hat{\mu}_{ho} = k_h - \hat{\mu}_{hn}, \quad \hat{\mu}_{lo} = k_a - \hat{\mu}_{ho} = k_a - k_h + \hat{\mu}_{hn}, \quad \text{and} \\
&\hat{\mu}_{ln} = k_l - \hat{\mu}_{lo} = k_v - k_a - \hat{\mu}_{hn}.
\end{aligned}$$

By substituting the above expressions of $\hat{\mu}_{lo}$ and $\hat{\mu}_{ln}$ in

$$\lambda_d \hat{\mu}_{lo} \hat{\mu}_{hn} + \rho_d \hat{\mu}_{hn} - \rho_u \hat{\mu}_{ln} = 0, \quad (\mu\text{-hn})$$

we obtain

$$\hat{\mu}_{hn} = \frac{1}{2}\left(\sqrt{\left(R + k_a - k_h\right)^2 + 4R \cdot k_h \left(1 - k_a\right)} - \left(R + k_a - k_h\right)\right), \tag{25}$$

where $R \equiv \frac{\rho_u + \rho_d}{\lambda_d}$. It is clear that $\hat{x} = (\hat{\mu}_{lo}, \hat{\mu}_{hn})$ solves the system $F(x; k_f) = 0$.

C.6.2 Comparative static derivatives of μ with respect to k_f

We apply Implicit Function Theorem. $F(x; k_f)$ is an infinitely differentiable function of $x \in \mathbb{R}^2$ and $k_f \in \mathbb{R}$, and the Jacobian matrix $\nabla_x F(\hat{x}; 0)$ is invertible (see Equation 13). As such, there is a differentiable function $x(k_f)$ defined in a neighborhood of $k_f = 0$ and $x = \hat{x}$ such that $F(x(k_f); k_f) = 0$. It is important to note that the derivative of $x(k_f)$ at any $k_f > 0$ is independent of the choice of the function $x(k_f)$; Claim 1 ensures that any choice of a function $x(k_f)$ gives the same value of x for each $k_f > 0$.

As explained above, the function $x(k_f)$, together with (11) and (12), defines $\mu(k_f) =$

$(\mu_i(k_f))_{i\in\mathcal{T}}$, which is also differentiable. Let $\mu_i' \equiv \frac{\partial\mu}{\partial k_f}\Big|_{k_f=0}$ for each $i \in \mathcal{T}$. Then,

$$\begin{aligned}
\begin{bmatrix}\mu_{lo}' \\ \mu_{hn}'\end{bmatrix} &= -\left[\nabla_x F(\hat{x};0)\right]^{-1} \frac{\partial F(\hat{x};0)}{\partial k_f} \\
&= -\begin{bmatrix}\lambda_d\hat{\mu}_{hn} + \lambda_f\hat{\mu}_{lo} + \rho_u & (\lambda_d - \lambda_f)\hat{\mu}_{lo} + \rho_d \\ (\lambda_d - \lambda_f)\hat{\mu}_{hn} + \rho_u & \lambda_d\hat{\mu}_{lo} + \lambda_f\hat{\mu}_{hn} + \rho_d\end{bmatrix}^{-1} \begin{bmatrix}\lambda_f\hat{\mu}_{lo} \\ 0\end{bmatrix}.
\end{aligned} \tag{26}$$

Also, $\mu_{ho}' = -\mu_{hn}'$ and $\mu_{ln}' = -\mu_{lo}'$.

From (11),

$$\begin{aligned}
\mu_{fn}' &= 1 - \mu_{hn}' + \mu_{lo}' \\
&= 1 - \frac{1}{det(\nabla_x F(\hat{x};0))} \begin{bmatrix}1 & -1\end{bmatrix} \begin{bmatrix}\lambda_d\hat{\mu}_{lo} + \lambda_f\hat{\mu}_{hn} + \rho_d & * \\ -(\lambda_d - \lambda_f)\hat{\mu}_{hn} - \rho_u & *\end{bmatrix} \begin{bmatrix}\lambda_f\hat{\mu}_{lo} \\ 0\end{bmatrix} \\
&= 1 - \frac{\lambda_f\hat{\mu}_{lo}\left(\lambda_d\hat{\mu}_{lo} + \lambda_d\hat{\mu}_{hn} + \rho_d + \rho_u\right)}{det(\nabla_x F(\hat{x};0))},
\end{aligned}$$

where

$$\begin{aligned}
det(\nabla_x F(\hat{x};0)) &= \left(\lambda_d\hat{\mu}_{hn} + \lambda_f\hat{\mu}_{lo} + \rho_u\right)\left(\lambda_d\hat{\mu}_{lo} + \lambda_f\hat{\mu}_{hn} + \rho_d\right) \\
&\quad - \left((\lambda_d - \lambda_f)\hat{\mu}_{hn} + \rho_u\right)\left((\lambda_d - \lambda_f)\hat{\mu}_{lo} + \rho_d\right) \\
&= \left(\lambda_d\hat{\mu}_{hn} + \rho_u\right)\left(\lambda_f\hat{\mu}_{hn} + \lambda_f\hat{\mu}_{lo}\right) + \left(\lambda_f\hat{\mu}_{hn} + \lambda_f\hat{\mu}_{lo}\right)\left(\lambda_d\hat{\mu}_{lo} + \rho_d\right) \\
&= \left(\lambda_f\hat{\mu}_{hn} + \lambda_f\hat{\mu}_{lo}\right)\left(\lambda_d\hat{\mu}_{lo} + \lambda_d\hat{\mu}_{hn} + \rho_d + \rho_u\right).
\end{aligned}$$

It follows that

$$\mu_{fn}' = \frac{\hat{\mu}_{hn}}{\hat{\mu}_{hn} + \hat{\mu}_{lo}} > 0. \tag{27}$$

Next, (12) implies $\mu_{fo} = \frac{(\lambda_f \mu_{hn} + \lambda_s \mu_{fn})(k_f - \mu_{fn})}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn} + \rho_e}$. Thus,

$$\begin{aligned}
\mu'_{fo} &= \frac{(1-\mu'_{fn})(\lambda_f \hat{\mu}_{hn})}{\rho_e + \lambda_f \hat{\mu}_{hn}} = \frac{\lambda_f \hat{\mu}_{hn} \hat{\mu}_{lo}}{(\rho_e + \lambda_f \hat{\mu}_{hn})(\hat{\mu}_{hn} + \hat{\mu}_{lo})} > 0, \\
\mu'_{fe} &= 1 - \mu'_{fn} - \mu'_{fo} = \frac{\rho_e \hat{\mu}_{lo}}{(\rho_e + \lambda_f \hat{\mu}_{hn})(\hat{\mu}_{hn} + \hat{\mu}_{lo})} > 0. \qquad (28)
\end{aligned}$$

C.6.3 Proof of Claim 2

From (16), $rv_{fn} = \lambda_f \mu_{lo} g_2 + \lambda_s \mu_{fe} g_3$, where g_2 and g_3 are determined by (17) and (20), respectively.

Let $v'_{fn} \equiv \left.\frac{\partial v_{fn}}{\partial k_f}\right|_{k_f=0}$ and $g'_m \equiv \left.\frac{\partial g_m}{\partial k_f}\right|_{k_f=0}$ for $m = 2, 3$. Then

$$rv'_{fn} = \lambda_f \left(\hat{g}_2 \mu'_{lo} + \hat{\mu}_{lo} g'_2\right) + \lambda_s \left(\hat{g}_3 \mu'_{fe} + \hat{\mu}_{fe} g'_3\right).$$

We find the value of each variable on the right-hand side of the above equation. For certain variables that we will use later, we remark whether the values are strictly positive and/or independent of λ_s.

We have observed the following properties:

1. (from (25)) the population $\hat{\mu} = (\mu_i)_{i \in \mathcal{T}}$ is strictly positive for corporate types, zero for fund types, and independent of λ_s,

2. (from (26), (27), and (28)) the derivative μ' is independent of λ_s,

3. (from (17)) As $g_3 = \frac{u_f - u_e}{2r + 2\rho_e + \lambda_f \mu_{hn} + \lambda_s \mu_{fn}}$, we have $\hat{g}_3 = \frac{u_f - u_e}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}} > 0$ and $g'_3 = -\frac{(\lambda_f \mu'_{hn} + \lambda_s \mu'_{fn}) \hat{g}_3}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}}$, which are independent of λ_s.

It remains to find the values of $\hat{g}_2$ and g'_2.

To state how g_2 is determined by (20), let $c_1 \equiv 2(r + \rho_u + \rho_d) + \lambda_d(\mu_{lo} + \mu_{hn})$, $c_2 \equiv 2\rho_u + \lambda_d \mu_{hn}$, and

$$D \equiv \begin{bmatrix} c_1 + \lambda_f(\mu_{fo} + \mu_{fe}) & c_1 + \lambda_f \mu_{fn} \\ c_2 - \lambda_f \mu_{hn} & c_2 + 2r + \lambda_f(\mu_{lo} + \mu_{fn}) \end{bmatrix}.$$

Also, let $\alpha_1 \equiv \frac{-D_{21}}{det(D)}$ and $\alpha_2 \equiv \frac{D_{11}}{det(D)}$. Then,

$$g_2 = \alpha_1 \left(u_h - u_l - \lambda_f \mu_{fe} g_3\right) + \alpha_2 \left(u_f - u_l - 2\rho_e g_3 - \lambda_s \mu_{fe} g_3\right).$$

Note that, when $k_f = 0$,

$$\begin{aligned}
&\hat{c}_1 = 2(r + \rho_u + \rho_d) + \lambda_d(\hat{\mu}_{lo} + \hat{\mu}_{hn}) > 0,\\
&\hat{c}_2 = 2\rho_u + \lambda_d \hat{\mu}_{hn} > 0,\\
&\hat{D} = \begin{bmatrix} \hat{c}_1 & \hat{c}_1 \\ \hat{c}_2 - \lambda_f \hat{\mu}_{hn} & \hat{c}_2 + 2r + \lambda_f \hat{\mu}_{lo} \end{bmatrix} \quad \text{(with a strictly positive determinant)},\\
&\hat{\alpha}_1 = \frac{-\hat{c}_2 + \lambda_f \hat{\mu}_{hn}}{det(\hat{D})} \text{ (the exact value is unnecessary for our proof)}, \quad \text{and}\\
&\hat{\alpha}_2 = \frac{\hat{c}_1}{det(\hat{D})} = \frac{1}{2r + \lambda_f(\hat{\mu}_{lo} + \hat{\mu}_{hn})} > 0,
\end{aligned} \tag{29}$$

which are all independent of λ_s. It follows that $\hat{g}_2 = \hat{\alpha}_1(u_h - u_l) + \hat{\alpha}_2(u_f - u_l - 2\rho_e \hat{g}_3)$ is independent of λ_s.

Last, let c_1', c_2', α_1', and α_2' be the corresponding variables' derivatives: e.g., $c_1' \equiv \frac{\partial c_1}{\partial k_f}\Big|_{k_f=0}$. The derivatives are all independent of λ_s, because $\hat{\mu}$ and μ' are independent of λ_s. Therefore,

$$\begin{aligned}
g_2' =& \alpha_1'(u_h - u_l - \lambda_f \hat{\mu}_{fe} \hat{g}_3) - \hat{\alpha}_1 \lambda_f (\mu_{fe}' \hat{g}_3 + \hat{\mu}_{fe} g_3')\\
&+ \alpha_2' \left(u_f - u_l - 2\rho_e \hat{g}_3 - \lambda_s \hat{\mu}_{fe} \hat{g}_3\right) - \hat{\alpha}_2 \left(2\rho_e g_3' + \lambda_s \mu_{fe}' \hat{g}_3 + \lambda_s \hat{\mu}_{fe} g_3'\right)\\
=& \alpha_1'(u_h - u_l) - \hat{\alpha}_1 \lambda_f \mu_{fe}' \hat{g}_3 + \alpha_2' \left(u_f - u_l - 2\rho_e \hat{g}_3\right) - \hat{\alpha}_2 \left(2\rho_e g_3' + \lambda_s \mu_{fe}' \hat{g}_3\right). \quad (\text{as } \hat{\mu}_{fe} = 0)
\end{aligned}$$

Only the last term $-\hat{\alpha}_2 \left(2\rho_e g_3' + \lambda_s \mu_{fe}' \hat{g}_3\right)$ is (affinely) dependent on λ_s, through $-\hat{\alpha}_2 \mu_{fe}' \hat{g}_3 \lambda_s$ and $g_3' = -\frac{(\lambda_f \mu_{hn}' + \lambda_s \mu_{fn}')\hat{g}_3}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}}$. As such, $g_2' = \gamma_1 \lambda_s + \gamma_2$, for $\gamma_1 = \hat{\alpha}_2 \hat{g}_3 \left(\frac{2\rho_e \mu_{fn}'}{2r + 2\rho_e + \lambda_f \hat{\mu}_{hn}} - \mu_{fe}'\right)$ and some γ_2 which aggregates all remaining terms. Both γ_1 and γ_2 are independent of λ_s.

Finally,

$$
\begin{aligned}
rv'_{fn} &= \lambda_f \left(\hat{g}_2\mu'_{lo} + \hat{\mu}_{lo}g'_2\right) + \lambda_s \left(\hat{g}_3\mu'_{fe} + \hat{\mu}_{fe}g'_3\right) \\
&= \lambda_f \left(\hat{g}_2\mu'_{lo} + \hat{\mu}_{lo}(\gamma_1\lambda_s + \gamma_2)\right) + \lambda_s\hat{g}_3\mu'_{fe} \quad (\text{as } \hat{\mu}_{fe} = 0) \\
&= (\lambda_f\hat{\mu}_{lo}\gamma_1 + \hat{g}_3\mu'_{fe})\lambda_s + (\lambda_f\hat{g}_2\mu'_{lo} + \lambda_f\hat{\mu}_{lo}\gamma_2),
\end{aligned}
$$

where the coefficient of λ_s and the last term are both independent of λ_s.

It remains to show that the coefficient of λ_s is strictly positive:

$$
\begin{aligned}
\lambda_f\hat{\mu}_{lo}\gamma_1 + \hat{g}_3\mu'_{fe} &= \lambda_f\hat{\mu}_{lo}\hat{\alpha}_2\hat{g}_3 \left(\frac{2\rho_e\mu'_{fn}}{2r + 2\rho_e + \lambda_f\hat{\mu}_{hn}} - \mu'_{fe}\right) + \hat{g}_3\mu'_{fe} \\
&> -\lambda_f\hat{\mu}_{lo}\hat{\alpha}_2\hat{g}_3\mu'_{fe} + \hat{g}_3\mu'_{fe} \quad (\text{as } \hat{\mu}_{lo}, \hat{\alpha}_2, \hat{g}_3, \mu'_{fn}, \hat{\mu}_{hn} \text{ are strictly positive}) \\
&= \mu'_{fe}\hat{g}_3 \left(1 - \lambda_f\hat{\alpha}_2\hat{\mu}_{lo}\right) \\
&= \mu'_{fe}\hat{g}_3 \left(1 - \frac{\lambda_f\hat{\mu}_{lo}}{2r + \lambda_f(\hat{\mu}_{lo} + \hat{\mu}_{hn})}\right) \quad (\text{from (29)}) \\
&> 0. \quad (\text{as } \mu'_{fe} \text{ and } \hat{g}_3 \text{ are strictly positive})
\end{aligned}
$$

C.7 Proof of Proposition 7

First, from (v-hn)-(v-ln),

$$
\begin{aligned}
rW_v \equiv & r(\mu_{ho}v_{ho} + \mu_{hn}v_{hn} + \mu_{lo}v_{lo} + \mu_{ln}v_{ln}) \\
= & \mu_{ho}(u_h + \rho_d(v_{lo} - v_{ho})) + \mu_{hn}(\lambda_d\mu_{lo}g_{lo-hn} + \lambda_f\mu_{fo}g_{fo-hn} + \lambda_f\mu_{fe}g_{fe-hn} + \rho_d\left(v_{ln} - v_{hn}\right)) \\
& + \mu_{lo}(u_l + \lambda_d\mu_{hn}g_{lo-hn} + \lambda_f\mu_{fn}g_{lo-fn} + \rho_u\left(v_{ho} - v_{lo}\right)) + \mu_{ln}\rho_u\left(v_{hn} - v_{ln}\right).
\end{aligned}
$$

We substitute $g_{fo-hn} = v_{ho} - v_{hn} - p_{fo-hn}$, $g_{fe-hn} = v_{ho} - v_{hn} - p_{fe-hn}$, $g_{lo-fn} = p_{lo-fn} - v_{lo} - v_{ln}$, and $g_{lo-hn} = (1/2)(v_{ho} + v_{ln} - v_{lo} + v_{hn})$. Then, (3) follows from

$$
\begin{aligned}
& rW_v - (\mu_{ho}u_h + \mu_{lo}u_l + \lambda_f\mu_{lo}\mu_{fn}p_{lo-fn} - \lambda_f\mu_{hn}(\mu_{fo}p_{fo-hn} + \mu_{fe}p_{fe-hn})) \\
& = (\rho_u\mu_{lo} - \rho_d\mu_{ho})(v_{ho} - v_{lo}) + (\rho_u\mu_{ln} - \rho_u\mu_{hn})(v_{hn} - v_{ln}) \\
& + \mu_{hn}(\lambda_d\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe})(v_{ho} - v_{hn}) + \mu_{lo}(\lambda_d\mu_{hn} + \lambda_f\mu_{fn})(v_{ln} - v_{lo}).
\end{aligned}
$$

The combined coefficient of v_{ho} on the right-hand side of the above equation is $-\rho_d\mu_{ho} + \rho_u\mu_{lo} + \mu_{hn}(\lambda_d\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe})$, which equals the right-hand side of the population equation (μ-ho), so it is zero. We can similarly verify that the combined coefficient of v_i for $i = hn, lo, ln$ are all equal to zero.

Second, we obtain (2) from all population equations (μ-hn)-(μ-fe) such that

$$\begin{aligned}
&rW - (\mu_{ho}u_h + \mu_{fo}u_f + \mu_{fe}u_e + \mu_{lo}u_l) \\
&= (\rho_u\mu_{lo} - \rho_d\mu_{ho})(v_{ho} - v_{lo}) + (\rho_u\mu_{ln} - \rho_d\mu_{hn})(v_{hn} - v_{ln}) + \rho_e\mu_{fo}(v_{fe} - v_{fo}) \\
&+ \mu_{hn}(\lambda_d\mu_{lo} + \lambda_f\mu_{fo} + \lambda_f\mu_{fe})(v_{ho} - v_{hn}) + \mu_{lo}(\lambda_d\mu_{hn} + \lambda_f\mu_{fn})(v_{ln} - v_{lo}) \\
&+ ((\lambda_f\mu_{lo} + \lambda_s\mu_{fe})\mu_{fn} - \lambda_f\mu_{hn}\mu_{fo})v_{fo} \\
&+ \lambda_f(\mu_{hn}\mu_{fo} + \mu_{hn}\mu_{fe} - \mu_{lo}\mu_{fn})v_{fn} - (\lambda_f\mu_{hn} + \lambda_s\mu_{fn})\mu_{fe}v_{fe}.
\end{aligned}$$

As before, we can verify that the combined coefficients of v_i for each $i \in \mathcal{T}$ equal the right-hand side of the type's population equation, so it is zero.

Lastly, the expression for W_f follows from $W_f = W - W_v$.

C.8 Proof of Proposition 9

First, consider the path of a lo-type investor in a steady-state equilibrium. This investor can sell its asset upon meeting either a buying investor (hn) or a fund buyer (fn). Each kind of meeting arrives with Poisson rate $\lambda_c\mu_{hn}$ or $\lambda_f\mu_{fn}$. The time until the first meeting of each kind, denoted by τ_{lo-hn} and τ_{lo-fn}, follows the exponential distributions. Thus, the time until selling $\tau_{sc} \equiv \min\{\tau_{lo-hn}, \tau_{lo-fn}\}$ follows an exponential distribution with parameter $\lambda_c\mu_{hn} + \lambda_f\mu_{fn}$. Hence, $E[\tau_{sc}] = \frac{1}{\lambda_c\mu_{hn}+\lambda_f\mu_{fn}}$.

Second, consider the path of a fo-type fund in a steady-state equilibrium. The fund sells its asset before receiving a liquidity shock to a buying investor (hn) or receives a liquidity shock and enters the exit phase (after which it can sell to either a buying investor (hn) or a fund buyer (fn)). We denote by τ_{fo} this period for which a fund maintains its type as fo. The time τ_{fo} follows an exponential distribution with parameter $\lambda_f\mu_{hn} + \rho_e$. Hence, $E[\tau_{fo}] = \frac{1}{\lambda_f\mu_{hn}+\rho_e}$.

Finally, we evaluate the path of an fe type fund (an outcome of an fo type fund receiving a liquidity shock before meeting a buying investor with probability $\frac{\rho_e}{\lambda_f \mu_{hn}+\rho_e}$). The fe type fund maintains its type until it sells its portfolio asset either to a buying investor (hn) or a fund buyer (fn). Thus, the fund maintains its type for the time period τ_{fe}, which follows an exponential distribution with parameter $\lambda_f \mu_{hn} + \lambda_s \mu_{fn}$. Hence, $E[\tau_{fe}] = \frac{1}{\lambda_f \mu_{hn}+\lambda_s \mu_{fn}}$.

As a result, the overall expected time for a fund to sell an asset is:

$$E[\tau_{sf}] = \frac{1}{\lambda_f \mu_{hn} + \rho_e} + \frac{\rho_e}{\lambda_f \mu_{hn} + \rho_e} \left(\frac{1}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn}} \right).$$

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Supplementary Appendix

In Appendix SA.1, we present the proofs of results related to the fast search market. In Appendix SA.2, we present proofs on spreads and prices.

SA.1 Proofs on the Fast Search Market

SA.1.1 Proof for Part 1 of Proposition 6

For any regular environment $\theta \equiv (k, r, u, \rho, \lambda)$, we consider a sequence $\theta^L \equiv (k, r, u, \rho, L\lambda)$ with $L \to \infty$. Let μ^L be the unique steady-state solution of $P(\theta^L)$ and v^L be the unique solution of $V(\theta^L)$ with $\mu(t)$ being replaced by μ^L.

In solving $P(\theta^L)$, it is more convenient to take $z \equiv 1/L$ and define another market $\psi^z \equiv (k, r, u, z\rho, \lambda)$. (i.e., low type-change rates, instead of high search rates) and solve $P(\psi^z)$. It is easy to verify that the unique steady-state solution μ^L of $P(\theta^L)$ also uniquely solves $P(\psi^z)$. Last, define $\psi^0 \equiv (k, r, u, 0, \lambda)$.

Lemma SA.1. *$\mu^0 \in \mathbb{R}^{\mathcal{T}}$ is a steady-state solution of $P(\psi^0)$ if and only if*

1. *(when $k_f + k_h > k_a$) $\mu^0_{ho} = \min\{k_a, k_h\}$, $\mu^0_{hn} = k_h - \mu^0_{ho}$, $\mu^0_{lo} = 0$, $\mu^0_{ln} = k_l$, $\mu^0_{fo} = \max\{0, k_a - k_h\}$, $\mu^0_{fe} = 0$, and $\mu^0_{fn} = k_f - \mu^0_{fo} - \mu^0_{fe}$.*

2. *(when $k_f + k_h < k_a$) $\mu^0_{ho} = k_h$, $\mu^0_{hn} = 0$, $\mu^0_{lo} = k_a - k_h - k_f$, $\mu^0_{ln} = k_l - \mu^0_{lo}$, $\mu^0_{fn} = 0$, and $\mu^0_{fo} + \mu^0_{fe} = k_f$.*

The problem $P(\psi^0)$ has multiple steady-state solutions in Case 2 ($k_f + k_h < k_a$), where many possible combinations of (μ_{fo}, μ_{fe}) satisfy $\mu_{fo} + \mu_{fe} = k_f$.

(Proof) The problem $P(\psi^0)$ consists of

$$(\lambda_d \mu_{lo} + \lambda_f \mu_{fo} + \lambda_f \mu_{fe})\mu_{hn} = 0, \qquad \text{(from (μ-ho))}$$

$$(\lambda_d \mu_{hn} + \lambda_f \mu_{fn})\mu_{lo} = 0, \qquad \text{(from (μ-ln))}$$

$$\lambda_f (\mu_{lo}\mu_{fn} - \mu_{hn}\mu_{fo}) + \lambda_s \mu_{fn}\mu_{fe} = 0, \qquad \text{(from (μ-fo))}$$

and the following four conditions that replace (μ-hn), (μ-lo), (μ-fe), and (μ-fn):

$$\mu_{ho} + \mu_{hn} = k_h, \quad \mu_{lo} + \mu_{ln} = k_l, \quad \mu_{ho} + \mu_{lo} + \mu_{fo} + \mu_{fe} = k_a, \quad \text{and} \quad \mu_{fn} + \mu_{fo} + \mu_{fe} = k_f.$$

It follows from $\lambda_d, \lambda_f, \lambda_d > 0$ that $\mu_{lo}\mu_{hn} = \mu_{fo}\mu_{hn} = \mu_{fe}\mu_{hn} = \mu_{lo}\mu_{fn} = \mu_{fn}\mu_{fe} = 0$.

Suppose that $k_f + k_h > k_a$. If $\mu_{lo} > 0$ or $\mu_{fe} > 0$, then $\mu_{hn} = 0$ and $\mu_{fn} = 0$, which results in a contradiction: $\mu_{ho} + (\mu_{fo} + \mu_{fe}) = k_h + k_f > k_a$. As $\mu_{lo} = \mu_{fe} = 0$, either $\mu_{fo} = 0$ or $\mu_{hn} = 0$. As $\mu_{ho} + \mu_{lo} + \mu_{fo} + \mu_{fe} = k_a > 0$, if $\mu_{fo} = 0$, then $\mu_{ho} = k_a$; for otherwise $\mu_{hn} = 0$ implies that $\mu_{ho} = k_h$ and $\mu_{fo} = k_a - k_h$. On the other hand, if $k_f + k_h < k_a$, then $\mu_{lo} > 0$, which implies that $\mu_{hn} = \mu_{fn} = 0$. Thus, $\mu_{ho} = k_h$, $\mu_{fo} + \mu_{fe} = k_f$, and $\mu_{lo} = k_a - k_h - k_f$. ■

The following lemma implies that $\lim_{L\to\infty} \mu^L$ exists in $\mathbb{R}_+^{\mathcal{T}}$:

Lemma SA.2. *There exists a solution μ^* of $P(\psi^0)$ such that $\mu^* \equiv \lim_{L\to\infty} \mu^L$.*

(Proof) For each $z \equiv 1/L$, let $F(\mu, \psi^z)$ denote the right-hand sides of the population equations (μ-hn)-(μ-fe) for a market $\psi^z \equiv (k, r, u, z\rho, \lambda)$. Define $f(\mu, z) \equiv -\|F(\mu, \psi^z)\|$, where $\|\cdot\|$ denotes the Euclidean norm. It is clear that μ^L with $L = 1/z$ is the unique maximizer of f with the maximum value equal to zero. Let $M(z) \equiv \{\mu^{1/z}\}$.

We similarly define $F(\mu, \psi^0)$ as the right-hand sides of the population equations for the market ψ^0 and $f(\mu, 0)$. Let $M(0)$ be the solution set of $\max_\mu f(\mu, 0)$. According to Lemma SA.1, the solution set $M(0)$ is singleton if $k_h + k_f > k_a$; for otherwise, $M(0)$ contains multiple solutions, each being different from others only in (μ_{fo}, μ_{fe}) under the constraint $\mu_{fo} + \mu_{fe} = k_f$.

The function f is continuous in μ and z because the equations F are continuous. Then, Berge's Maximum Theorem implies that $M(\cdot)$ is upper hemicontinuous at $z = 0$:

1. (when $k_h + k_f > k_a$) μ^L converges to the unique solution of $P(\psi^0)$.

2. (when $k_h + k_f < k_a$) for each type $i \neq fo, fe$, the population μ_i^L converges to μ_i^0 given in Lemma SA.1, and $\mu_{fo}^L + \mu_{fe}^L$ converges to k_f.

It remains to show that, when $k_h + k_f > k_a$, the sequence μ_{fe}^L converges. (The convergence of μ_{fo}^L follows immediately from $\lim_{L\to\infty}(\mu_{fo}^L + \mu_{fe}^L) = k_f$.)

For every $L > 0$, $L(\lambda_f \mu^L_{hn} + \lambda_s \mu^L_{fn})\mu^L_{fe} = \rho_e(k_f - \mu^L_{fn} - \mu^L_{fe})$ (from (μ-fe)), which implies

$$\mu^L_{fe} = \frac{\rho_e(k_f - \mu^L_{fn})}{\rho_e + L(\lambda_f \mu^L_{hn} + \lambda_s \mu^L_{fn})}. \tag{SA.1}$$

We find $\lim_{L\to\infty} L(\lambda_f \mu^L_{hn} + \lambda_s \mu^L_{fn}$ from

$$\begin{aligned} &L\mu^L_{hn}(\lambda_d \mu^L_{lo} + \lambda_f \mu^L_{fo} + \lambda_f \mu^L_{fe}) = -\rho_d \mu^L_{hn} + \rho_u \mu^L_{ln}, && \text{(from (μ-hn))} \\ &L(\lambda_v \mu^L_{hn} + \lambda_f \mu^L_{fn})\mu^L_{lo} = -\rho_u \mu^L_{lo} + \rho_d \mu^L_{ho}. && \text{(from (μ-lo))} \end{aligned}$$

By the convergence of μ^L_i for $i \neq fo, fe$, and the convergence of $\mu^L_{fe} + \mu^L_{fo}$ to n_f,

$$\begin{aligned} &\lim_{L\to\infty} L\mu^L_{hn} = \frac{\rho_u \mu^*_{ln}}{\lambda_d \mu^*_{lo} + \lambda_f k_f}, \quad \text{and} \\ &\lim_{L\to\infty} L(\lambda_v \mu^L_{hn} + \lambda_f \mu^L_{fn}) = \frac{\rho_d \mu^*_{ho} - \rho_u \mu^*_{lo}}{\mu^*_{lo}} = \frac{\rho_d k_h - \rho_u \mu^*_{lo}}{\mu^*_{lo}} = \frac{\rho_u \mu^*_{ln}}{\mu^*_{lo}}. \end{aligned}$$

It follows that

$$\begin{aligned} \lim_{L\to\infty} L(\lambda_f \mu^L_{hn} + \lambda_s \mu^L_{fn}) &= \frac{\lambda_s \rho_u \mu^*_{ln}}{\lambda_f \mu^*_{lo}} + \left(\lambda_f - \frac{\lambda_d \lambda_s}{\lambda_f}\right) \frac{\rho_u \mu^*_{ln}}{\lambda_d \mu^*_{lo} + \lambda_f k_f} \\ &= \frac{\rho_u \mu^*_{ln}}{\mu^*_{lo}} \frac{\lambda_f \mu^*_{lo} + \lambda_s k_f}{\lambda_d \mu^*_{lo} + \lambda_f k_f} > 0. \end{aligned}$$

Therefore,

$$\mu^*_{fe} \equiv \lim_{L\to\infty} \mu^L_{fe} = \frac{k_f}{1 + \frac{\rho_u \mu^*_{ln}}{\rho_e \mu^*_{lo}} \frac{\lambda_f \mu^*_{lo} + \lambda_s k_f}{\lambda_d \mu^*_{lo} + \lambda_f k_f}}, \quad \text{and} \quad \mu^*_{fo} = k_f - \mu^*_{fe}.$$

■

SA.1.2 Proof for Part 2 of Proposition 6

We divide the proof into two lemmas.

Lemma SA.3. *For every $i \in \mathcal{T}$, if $\mu^*_i = 0$, then $\mu^{**}_i \equiv \lim_{L\to\infty} L\mu^L_i$ exists in $\mathbb{R}$.*

(Proof) The following table summarizes the population limits for some types from

Lemma SA.1 and Lemma SA.2:

	A. $k_a < k_h$	B. $k_h < k_a < k_h + k_f$	C. $k_h + k_f < k_a$
$\mu^*_{ho} =$	k_a	k_h	k_h
$\mu^*_{fo} =$	0	$k_a - k_h$	$< k_f$
$\mu^*_{lo} =$	0	0	$k_a - k_f - k_h$
$\mu^*_{fe} =$	0	0	> 0

As μ^*_{ho} and μ^*_{ln} are always strictly positive, the following step considers only other types.

Suppose $\mu^*_{hn} = 0$ (Cases A, B, and C): for any L,

$$L(\lambda_c \mu^L_{lo} + \lambda_f \mu^L_{fo} + \lambda_f \mu^L_{fe})\mu^L_{hn} = -\rho_d \mu^L_{hn} + \rho_u \mu^L_{ln}. \quad \text{(from (μ-hn))}$$

By Lemma 1, $\mu^L_i > 0$ for every $i \in \mathcal{T}$,

$$L\mu^L_{hn} = \frac{\rho_u \mu^L_{ln} - \rho_d \mu^L_{hn}}{\lambda_d \mu^L_{lo} + \lambda_f \mu^L_{fo} + \lambda_f \mu^L_{fe}} \tag{SA.2}$$

$$\implies \mu^{**}_{hn} \equiv \lim_{L\to\infty} L\mu^L_{hn} = \frac{\rho_u \mu^*_{ln} - \rho_d \mu^*_{hn}}{\lambda_d \mu^*_{lo} + \lambda_f(\mu^*_{fo} + \mu^*_{fe})} = \frac{\rho_u \mu^*_{ln}}{\lambda_d \mu^*_{lo} + \lambda_f(\mu^*_{fo} + \mu^*_{fe})} > 0. \tag{SA.3}$$

Suppose $\mu^*_{lo} = 0$ (Cases A and B): for every L,

$$(\lambda_d \mu^L_{hn} + \lambda_f \mu^L_{fn})(L\mu^L_{lo}) = \rho_d \mu^L_{ho} - \rho_u \mu^L_{lo}. \quad \text{(from (μ-lo))}$$

It follows that

$$\mu^{**}_{lo} \equiv \lim_{L\to\infty} L\mu^L_{lo} = \frac{\rho_d \mu^*_{ho} - \rho_u \mu^*_{lo}}{\lambda_d \mu^*_{hn} + \lambda_f \mu^*_{fn}} = \frac{\rho_d \mu^*_{ho}}{\lambda_d \mu^*_{hn} + \lambda_f \mu^*_{fn}}.$$

Suppose $\mu^*_{fe} = 0$ (Cases A and B): for every L,

$$(\lambda_f \mu^L_{hn} + \lambda_s \mu^L_{fn})(L\mu^L_{fe}) = \rho_e \mu^L_{fo}. \quad \text{(from (μ-fe))}$$

It follows that

$$\mu^{**}_{fe} \equiv \lim_{L\to\infty} L\mu^L_{fe} = \frac{\rho_e \mu^*_{fo}}{\lambda_f \mu^*_{hn} + \lambda_s \mu^*_{fn}}.$$

Suppose $\mu^*_{fo} = 0$ (Case A): for every L,

$$L(\lambda_f \mu^L_{lo} + \lambda_s \mu^L_{fe})\mu^L_{fn} = L\lambda_f \mu^L_{hn}\mu^L_{fo} + \rho_e \mu^L_{fo}. \quad \text{(from (μ-fo))}$$

It follows from the convergence of $L\mu^L_{lo}$ and $L\mu^L_{fe}$ in Case A that

$$\mu^{**}_{fo} \equiv \lim_{L\to\infty} L\mu^L_{fo} = \lim_{L\to\infty} \frac{(\lambda_f(L\mu^L_{lo}) + \lambda_s(L\mu^L_{fe}))\mu^L_{fn} - \rho_e \mu^L_{fo}}{\lambda_f \mu^L_{hn}} = \frac{(\lambda_f \mu^{**}_{lo} + \lambda_s \mu^{**}_{fe})\mu^*_{fn}}{\lambda_f \mu^*_{hn}}.$$

Finally, suppose $\mu^*_{fn} = 0$ (Case C): for every L,

$$\mu^L_{hn}\mu^L_{fo} + \mu^L_{hn}\mu^L_{fe} = \mu^L_{lo}\mu^L_{fn}. \quad \text{(from (μ-fn))}$$

As $\mu^L_{lo} > 0$ (Lemma 1), we have

$$L\mu^L_{fn} = \frac{L\mu^L_{hn}(\mu^L_{fo} + \mu^L_{fe})}{\mu^L_{lo}}. \tag{SA.4}$$

It follows from the convergences of $L\mu^L_{hn}$ that

$$\mu^{**}_{fn} \equiv \lim_{L\to\infty} L\mu^L_{fn} = \lim_{L\to\infty} \frac{L\mu^L_{hn}(\mu^L_{fo} + \mu^L_{fe})}{\mu^L_{lo}} = \frac{\mu^{**}_{hn}(\mu^*_{fo} + \mu^*_{fe})}{\mu^*_{lo}} > 0. \tag{SA.5}$$

■

Lemma SA.4. *For any $i \in \mathcal{T}$, if $\mu^*_i > 0$, then $\mu^{**}_i \equiv \lim_{L\to\infty} L(\mu^L_i - \mu^*_i)$ exists in $\mathbb{R}$.*

(Proof)

First, consider Case A ($k_a < k_h$): Only $\mu^*_{ho}, \mu^*_{fn}, \mu^*_{ln}$ are strictly positive. As $L \to \infty$,

$$L(\mu^L_{ho} - \mu^*_{ho}) = L(k_a - \mu^L_{lo} - \mu^L_{fo} - \mu^L_{fe}) - L(k_a - \mu^*_{lo} - \mu^*_{fo} - \mu^*_{fe}) \to -\mu^{**}_{lo} - \mu^{**}_{fo} - \mu^{**}_{fe},$$

where the convergence of $L\mu^L_{lo}$, $L\mu^L_{fo}$, and $L\mu^L_{fe}$ holds by Lemma SA.3.

We similarly find the convergence speed for μ_{fn}^L and μ_{ln}^L:

$$\begin{aligned} L(\mu_{fn}^L - \mu_{fn}^*) &= L(k_f - \mu_{fo}^L - \mu_{fe}^L) - L(k_f - \mu_{fo}^* - \mu_{fe}^*) \to -\mu_{fo}^{**} - \mu_{fe}^{**}, \quad \text{and} \\ L(\mu_{ln}^L - \mu_{ln}^*) &= L(k_l - \mu_{ln}^L) - L(k_l - \mu_{lo}^*) \to -\mu_{lo}^{**}. \end{aligned}$$

Next, consider Case B ($k_h < k_a < k_h + k_f$):
Only $\mu_{ho}^*, \mu_{fo}^*, \mu_{ln}^*$, and μ_{fn}^* are strictly positive. As $L \to \infty$,

$$\begin{aligned} L(\mu_{ho}^L - \mu_{ho}^*) &= L(k_h - \mu_{hn}^L) - L(k_h - \mu_{hn}^*) \to -\mu_{hn}^{**}, \\ L(\mu_{ln}^L - \mu_{ln}^*) &= L(k_l - \mu_{lo}^L) - L(k_l - \mu_{lo}^*) \to -\mu_{lo}^{**}, \\ L(\mu_{fo}^L - \mu_{fo}^*) &= L(\mu_{fo}^L - (k_a - k_h)) = -L\mu_{lo}^L - L\mu_{fe}^L - L(\mu_{ho}^L - k_h) \\ &\to -\mu_{lo}^{**} - \mu_{fe}^{**} + \mu_{hn}^{**}, \quad \text{and} \\ L(\mu_{fn}^L - \mu_{fn}^*) &= L(\mu_{fn}^L - (k_f - k_a + k_h)) = -L\mu_{fe}^L - L(\mu_{fo}^L - (k_a - k_h)) \\ &\to -\mu_{fe}^{**} + (\mu_{lo}^{**} + \mu_{fe}^{**} - \mu_{hn}^{**}). \end{aligned}$$

Finally, in Case C ($k_h + k_f < k_a$), we have $\mu_{ho}^*, \mu_{lo}^*, \mu_{ln}^*, \mu_{fo}^*$, and μ_{fe}^* that are strictly positive. The proof for the first three types are similar to the previous cases: as $L \to \infty$,

$$\begin{aligned} L(\mu_{ho}^L - \mu_{ho}^*) &= L(k_h - \mu_{hn}^L) - L(k_h - \mu_{hn}^*) \to -\mu_{hn}^{**}, \\ L(\mu_{lo}^L - \mu_{lo}^*) &= L(\mu_{lo}^L - (k_a - k_h - k_f)) = -L(\mu_{fo}^L + \mu_{fe}^L - k_f) - L(\mu_{ho}^L - k_h) \\ &\to -\mu_{fn}^{**} + \mu_{hn}^{**}, \qquad \text{(SA.6)} \\ L(\mu_{ln}^L - \mu_{ln}^*) &= L(k_l - \mu_{lo}^L) - L(k_l - \mu_{lo}^*) \to -\mu_{lo}^{**} = \mu_{fn}^{**} - \mu_{hn}^{**}. \qquad \text{(SA.7)} \end{aligned}$$

It remains to show the convergence speed for μ_{fo}^L and μ_{fe}^L. On the one hand, from (μ-fe) and the convergence of μ_{fe}^L, μ_{fo}^L, $L\mu_{hn}^L$, and $L\mu_{fn}^L$, we have

$$L(\lambda_f \mu_{hn}^L + \lambda_s \mu_{fn}^L)\mu_{fe}^L = \rho_e \mu_{fo}^L \quad \text{and} \quad (\lambda_f \mu_{hn}^{**} + \lambda_s \mu_{fn}^{**})\mu_{fe}^* = \rho_e \mu_{fo}^*.$$

Let $\phi^L \equiv L(\lambda_f \mu_{hn}^L + \lambda_s \mu_{fn}^L)$, and $\phi^{**} \equiv \lambda_f \mu_{hn}^{**} + \lambda_s \mu_{fn}^{**}$. Then,

$$\rho_e L(\mu_{fo}^L - \mu_{fo}^*) = \phi^L L\mu_{fe}^L - \phi^{**} L\mu_{fe}^* = L(\phi^L - \phi^{**})\mu_{fe}^L + \phi^{**} L(\mu_{fe}^L - \mu_{fe}^*). \qquad \text{(SA.8)}$$

On the other hand, from $\mu_{fn}^L + \mu_{fo}^L + \mu_{fe}^L = k_f$ and $\mu_{fo}^* + \mu_{fe}^* = k_f$, we have

$$L(\mu_{fo}^L - \mu_{fo}^*) + L(\mu_{fe}^L - \mu_{fe}^*) = -L\mu_{fn}^L. \tag{SA.9}$$

By summarizing (SA.8) and (SA.9), for every L,

$$\begin{bmatrix} L(\mu_{fo}^L - \mu_{fo}^*) \\ L(\mu_{fe}^L - \mu_{fe}^*) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \rho_e & -\phi^{**} \end{bmatrix}^{-1} \begin{bmatrix} -L\mu_{fn}^L \\ L(\phi^L - \phi^{**})\mu_{fe}^L \end{bmatrix},$$

where the inverse matrix is well-defined because $\phi^{**} > 0$ (see (SA.3) and (SA.5)). Note that $L\mu_{fn}^L$ and μ_{fe}^L converge (see (SA.5) and Lemma SA.2). It remains to prove that

$$L(\phi^L - \phi^{**}) = \lambda_f L(L\mu_{hn}^L - \mu_{hn}^{**}) + \lambda_s L(L\mu_{fn}^L - \mu_{fn}^{**}) \quad \text{converges as } L \to \infty.$$

From (SA.2), (SA.3), $\mu_{hn}^* = 0$, and $\mu_{fo}^* + \mu_{fe}^* = k_f$,[38] we have

$$L(L\mu_{hn}^L - \mu_{hn}^{**}) = \frac{\rho_u L\mu_{ln}^L - \rho_d L\mu_{hn}^L}{\lambda_d \mu_{lo}^L + \lambda_f(\mu_{fo}^L + \mu_{fe}^L)} - \frac{\rho_u (L\mu_{ln}^*)}{\lambda_d \mu_{lo}^* + \lambda_f k_f}.$$

To ease expositions, let A^L and A^* denote the denominators in the above equation. Then,

$$\begin{aligned} L(L\mu_{hn}^L - \mu_{hn}^{**}) &= \frac{\rho_u L\mu_{ln}^L - \rho_d L\mu_{hn}^L}{A^L} - \frac{\rho_u L\mu_{ln}^*}{A^*} \\ &= \frac{\rho_u L(\mu_{ln}^L - \mu_{ln}^*) - \rho_d L\mu_{hn}^L}{A^L} + \rho_u \mu_{ln}^* L\left(\frac{1}{A^L} - \frac{1}{A^*}\right) \\ &= \frac{\rho_u L(\mu_{ln}^L - \mu_{ln}^*) - \rho_d L\mu_{hn}^L}{A^L} - \rho_u \mu_{ln}^* \frac{\lambda_d L(\mu_{lo}^L - \mu_{lo}^*) - \lambda_f L\mu_{fn}^L}{A^L A^*}, \end{aligned} \tag{SA.10}$$

which converges by (SA.3), (SA.5), (SA.6), and (SA.7).

Then, from (SA.4), (SA.5), and $\mu_{fo}^* + \mu_{fe}^* = k_f$, we have

$$L(L\mu_{fn}^L - \mu_{fn}^{**}) = \frac{L\mu_{hn}^L L(\mu_{fo}^L + \mu_{fe}^L)}{\mu_{lo}^L} - \frac{\mu_{hn}^{**} L k_f}{\mu_{lo}^*}.$$

[38]Recall that we are considering Case C ($k_h + k_f < k_a$).

Then,

$$
\begin{aligned}
L(L\mu_{fn}^L - \mu_{fn}^{**}) &= \frac{L\mu_{hn}^L L(\mu_{fo}^L + \mu_{fe}^L - k_f)}{\mu_{lo}^L} + \frac{L^2\mu_{hn}^L k_f}{\mu_{lo}^L} - \frac{L\mu_{hn}^{**}k_f}{\mu_{lo}^*} \\
&= -\frac{(L\mu_{hn}^L)(L\mu_{fn}^L)}{\mu_{lo}^L} + \frac{L(L\mu_{hn}^L - \mu_{hn}^{**})k_f}{\mu_{lo}^L} + \frac{L\mu_{hn}^{**}k_f}{\mu_{lo}^L} - \frac{L\mu_{hn}^{**}k_f}{\mu_{lo}^*} \\
&= -\frac{(L\mu_{hn}^L)(L\mu_{fn}^L)}{\mu_{lo}^L} + \frac{L(L\mu_{hn}^L - \mu_{hn}^{**})k_f}{\mu_{lo}^L} - \frac{\mu_{hn}^{**}k_f L(\mu_{lo}^L - \mu_{lo}^*)}{\mu_{lo}^L\mu_{lo}^*},
\end{aligned}
$$

which converges by (SA.3), (SA.5), (SA.6), and (SA.10). ■

SA.1.3 Proof of Proposition 8

By Lemma SA.1 and Lemma SA.2, if $k_a < k_h + k_f$, then $\mu_{ho}^* = \min\{k_a, k_h\}$, $\mu_{fo}^* = \max\{0, k_a - k_h\}$, $\mu_{fe}^* = 0$, and $\mu_{lo}^* = 0$. Since μ^* coincides with the efficient asset allocation $\overline{\mu}$, we have $W^* = \overline{W}$. The independence of W^* on u_f and u_e is trivial as $\mu_{fo}^* = \mu_{fe}^* = 0$. The independence of W^* on λ_d also follows from $\overline{W}$'s independence of any search friction. When $k_h < k_a < k_h + k_f$, we have $\mu_{fo} > 0$, so $W^* = \overline{W}$ is strictly increasing in u_f.

If $k_a > k_h + k_f$, then $rW^* = r\overline{W} - \mu_{fe}^*(u_f - u_e)$. We have $W^* < \overline{W}$ because

$$
\mu_{fe}^* = \frac{k_f}{1 + \frac{\rho_u\mu_{ln}^*}{\rho_e\mu_{lo}^*}\frac{\lambda_f\mu_{lo}^* + \lambda_s k_f}{\lambda_d\mu_{lo}^* + \lambda_f k_f}} > 0,
$$

The welfare W^* is increasing in u_f and u_e as μ_{fo}^* and μ_{fe}^* are strictly positive. Moreover, μ_{fe}^* is decreasing in λ_s and increasing in λ_d. Thus, the welfare W^* is increasing in λ_s and decreasing in λ_d.

SA.2 Proofs on Spreads and Prices

SA.2.1 Proof of Proposition 11

It remains to obtain the closed-form expression of PV_{calls}. The time taken to buy τ_b, the time taken to sell τ_s, and the event of purchasing from a low-type investor, rather than an exiting fund, are all independent from each other. Thus,

$$PV_{\text{calls}} = E\left[P_b\right] E\left[e^{-r\tau_b}\right] + E\left[(fP_b)\right] E\left[\int_0^{\tau_b+\tau_s} e^{-rt}dt\right],$$

where

$$\begin{aligned} E\left[\int_0^{\tau_b+\tau_s} e^{-rt}dt\right] &= E\left[\int_0^{\tau_b} e^{-rt}dt\right] + E\left[\int_{\tau_b}^{\tau_b+\tau_s} e^{-rt}dt\right] \\ &= E\left[\int_0^{\tau_b} e^{-rt}dt\right] + E\left[e^{-r\tau_b}\right] E\left[\int_0^{\tau_s} e^{-rt}dt\right]. \end{aligned}$$

Note that

$$\begin{aligned} E\left[P_b\right] &= \frac{\lambda_f \mu_{lo} P_{lo-fn} + \lambda_s \mu_{fe} P_{fe-fn}}{\lambda_f \mu_{lo} + \lambda_s \mu_{fe}}, \\ E\left[e^{-r\tau_b}\right] &= \frac{\lambda_f \mu_{lo} + \lambda_s \mu_{fe}}{\lambda_f \mu_{lo} + \lambda_s \mu_{fe} + r}, \quad \text{and} \\ E\left[\int_0^{\tau_b} e^{-rt}dt\right] &= \frac{1}{\lambda_f \mu_{lo} + \lambda_s \mu_{fe} + r}. \end{aligned}$$

Last, recall that a fund's type remains fo or fe for the time period $\tau_{fo} \equiv \min\{\tau_{fo-hn}, \tau_e\}$ or $\tau_e \equiv \min\{\tau_{fe-hn}, \tau_{fe-fn}\}$, respectively. Then,

$$\begin{aligned} E\left[\int_0^{\tau_s} e^{-rt}dt\right] &= E\left[\int_0^{\tau_{fo}} e^{-rt}dt\right] + E\left[\mathbf{1}_{\tau_{fo}=\tau_e}\right] E\left[e^{-r\tau_{fo}}\right] E\left[\int_0^{\tau_e} e^{-rt}dt\right] \\ &= \frac{1}{\lambda_f \mu_{hn} + \rho_e + r} + \frac{\rho_e}{\lambda_f \mu_{hn} + \rho_e} \frac{\lambda_f \mu_{hn} + \rho_e}{\lambda_f \mu_{hn} + \rho_e + r} \frac{1}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn} + r}. \end{aligned}$$

It follows that

$$PV_{\text{calls}} = \frac{\lambda_f \mu_{lo} P_{lo-fn} + \lambda_s \mu_{fe} P_{fe-fn}}{\lambda_f \mu_{lo} + \lambda_s \mu_{fe} + r} \left(1 + f\left(\frac{1}{\lambda_f \mu_{lo} + \lambda_s \mu_{fe}} + \frac{1 + \rho_e \left(\frac{1}{\lambda_f \mu_{hn} + \lambda_s \mu_{fn} + r}\right)}{\lambda_f \mu_{hn} + \rho_e + r}\right)\right).$$

SA.2.2 Proof of Proposition 12

For Part 1:
$$2(p_{fo-hn} - p_{fe-hn}) = v_{fo} - v_{fe} = 2g_{fe-fn} \geq 0,$$
$$2(p_{fe-hn} - p_{fe-fn}) = (v_{ho} - v_{hn}) - (v_{fo} - v_{fn}) = 2g_{fo-hn} \geq 0.$$

For Part 2:
$$\begin{aligned} 2(p_{fo-hn} - p_{lo-hn}) &= (v_{ho} - v_{hn} + v_{fo} - v_{fn}) - (v_{lo} - v_{ln} + v_{fo} - v_{fn}) \\ &= (v_{ho} - v_{hn}) - (v_{lo} - v_{ln}) = 2g_{lo-hn} \geq 0, \\ 2(p_{lo-hn} - p_{lo-fn}) &= (v_{ho} - v_{hn}) - (v_{fo} - v_{fn}) = 2g_{fo-hn} \geq 0. \end{aligned}$$